AweSums:

The Majesty of Mathematics

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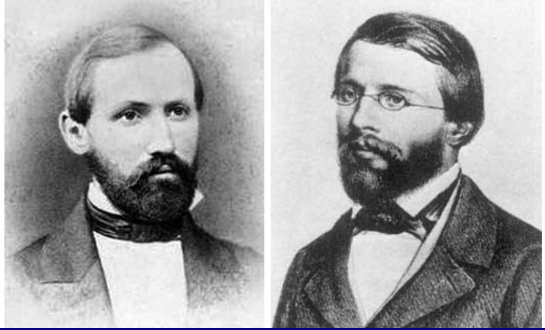
Evening Course, UCD, Autumn 2016

Outline

Outline

AweSums: The Majesty of Maths



Bernhard Riemann (1826-66)

AweSums: The Majesty of Maths

We aim to get a flavour of the Riemann Hypothesis.

It involves the zeros of the “Zeta function”:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

So, we need to talk about several new topics.

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It involves the zeros of the "Zeta function":

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

So, we need to talk about several new topics.

In this lecture, we will look at Euler's Product-Sum Formula, the Prime Number Theorem and the RH.

Outline

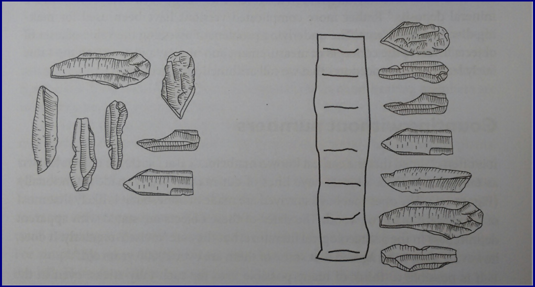
Lecture Content

1. Emergence of Numbers.
2. Georg Cantor. Set Theory.
3. Pythagoras. Irrational Numbers.
4. Hilbert. Gauss. The Real Number Line.
5. Powers. Logarithms. Prime Numbers.
6. Functions. Archimedes. Natural Logs.
7. Exponential Growth. Euler. Sequences & Series.
8. Trigonometry. Taylor Series.
9. Basel Problem. Complex Numbers. Euler's Formula.
10. Prime Number Theorem and Riemann Hypothesis.

Lecture 1

Emergence of Numbers.

Keeping Stock without Counting

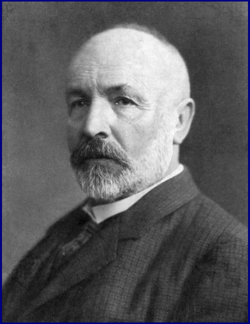


- Fair division of food
- Counting animals
- Measuring land
- etc., etc.

Lecture 2

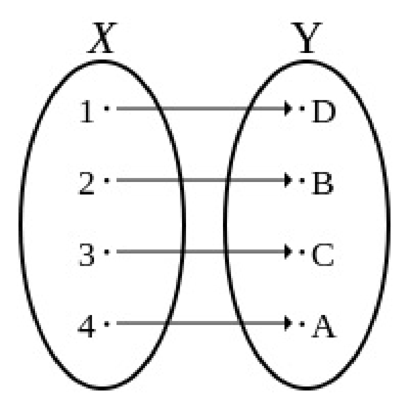
Georg Cantor. Set Theory.

Georg Cantor (1845–1918)



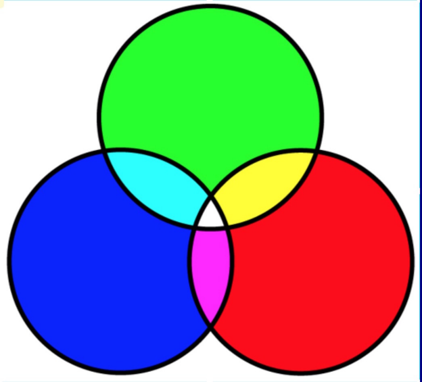
Cantor discovered many remarkable properties of infinite sets.

One-to-one Correspondence



The Intersection of 3 Sets

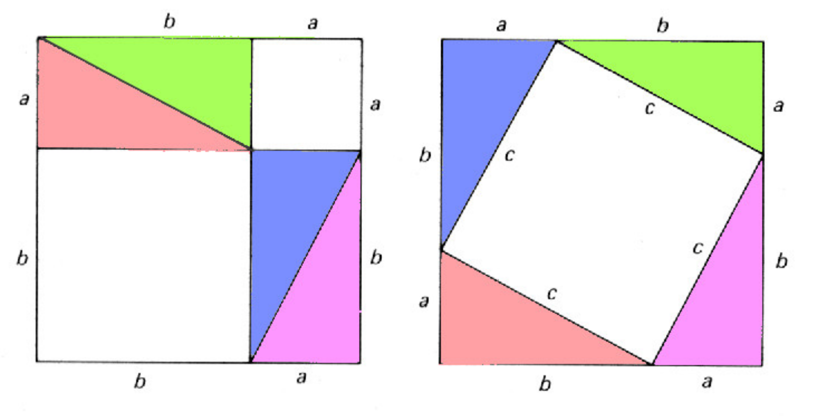
The three overlapping circles have attained an iconic status, seen in a huge range of contexts.



Lecture 3

Pythagoras. Irrational Numbers.

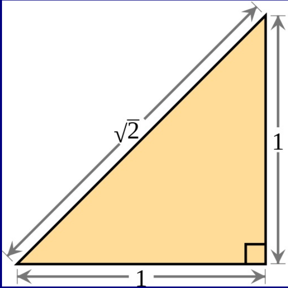
Proof without Formulae



$$a^2 + b^2 = c^2$$

Irrational Numbers

If the side of a square is of length 1, then the diagonal has length $\sqrt{2}$ (by the Theorem of Pythagoras).



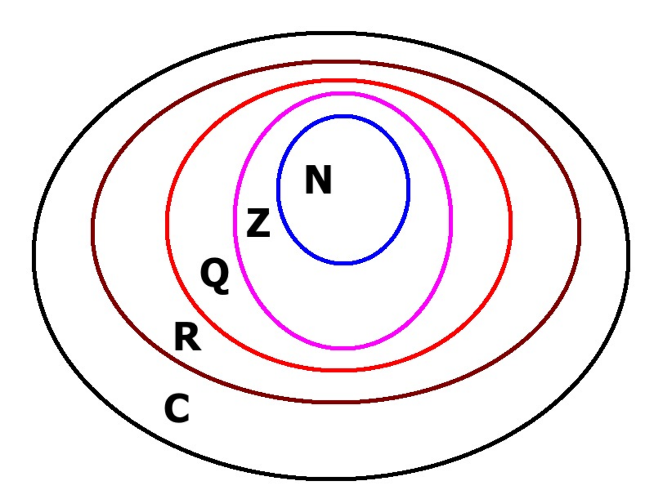
The ratio between the diagonal and the side is:

$$\frac{\text{Diagonal}}{\text{Side Length}} = \sqrt{2}$$

Lecture 4

Hilbert. Gauss. The Real Number Line.

The Hierarchy of Numbers



$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

Carl Friedrich Gauss (1777–1855)

A German mathematician who made profound contributions to many fields of mathematics:

- Number theory
- Algebra
- Statistics
- Analysis
- Differential geometry
- Geodesy & Geophysics
- Mechanics & Electrostatics
- Astronomy

Gauss is regarded as one of the greatest mathematicians of all time.

Lecture 5

Powers. Logarithms. Prime Numbers.

Multiplying Powers

We can denote large numbers by using **exponents:**

3×3	is written	3^2
$5 \times 5 \times 5$	is written	5^3
$7 \times 7 \times 7 \times 7$	is written	7^4

Thus,

$$7^2 \times 7^3 = 7^{2+3}$$

Note: Multiply on left. Add on right.

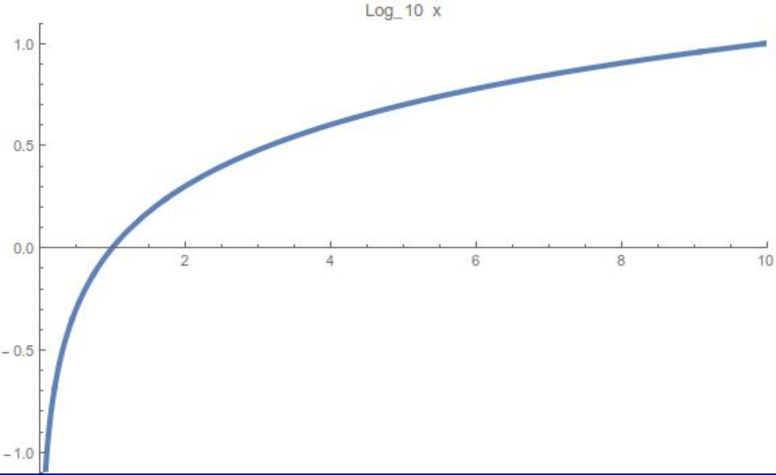
Definition of Log_{10} x

DEFINITION: The logarithm of x is the power to which 10 must be raised to give x:

$$\log_{10} y = x \quad \Longleftrightarrow \quad 10^x = y$$

$100 = 10^2$	**so that**	$\log_{10} 100 = 2$
$1\ 000 = 10^3$	**so that**	$\log_{10} 1\ 000 = 3$
$1\ 000\ 000 = 10^6$	**so that**	$\log_{10} 1\ 000\ 000 = 6$

$\text{Log}_{10}\, x$ for $0 < x < 10$



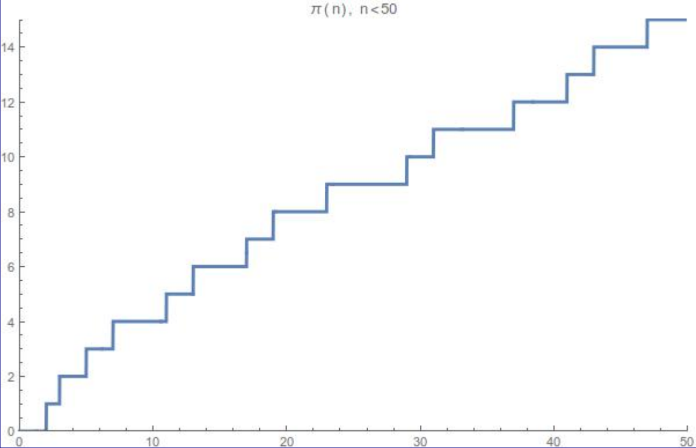


Figure : The prime counting function $\pi(n)$ for $0 \le n \le 50$.

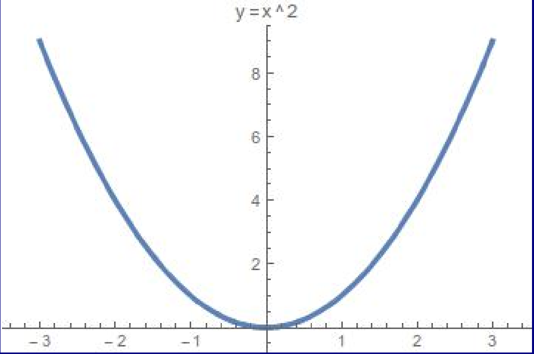
Lecture 6

Functions. Archimedes. Natural Logs.

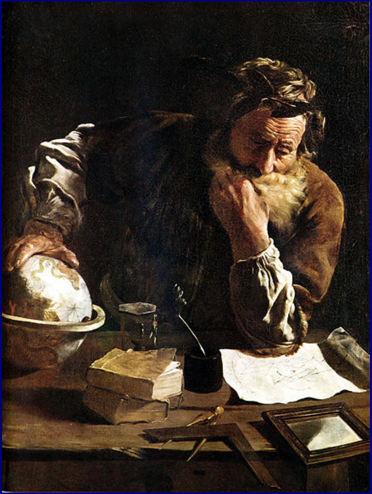
Function Defined by a Graph

The set of all (input, output) pairs is called the graph:

$$G = \{(x, x^2) : x \in [-3, +3]\}$$

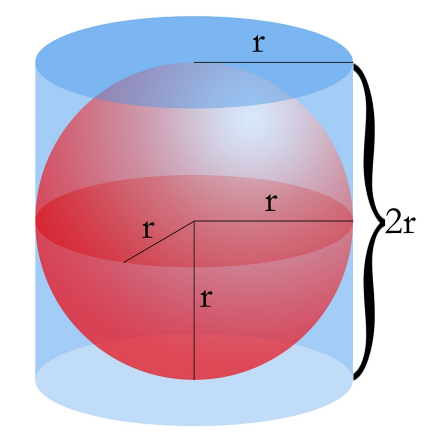


$A\rho\chi\iota\mu\eta\delta\eta\varsigma$



Archimedes Thoughtful **by Domenico Fetti (1620)**

Archimedes Great Discovery



Volume of Cylinder:

$$V_C = \pi r^2 \times 2r$$

Volume of Sphere:

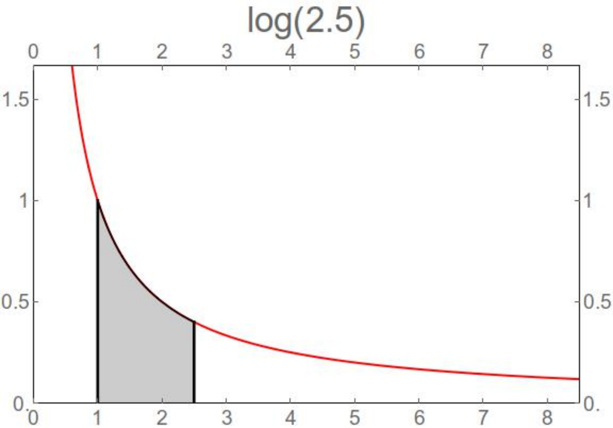
$$V_S = \frac{2}{3} V_C$$

Therefore

$$V_S = \frac{4}{3} \pi r^3$$

Definition of Natural Logarithm

The natural log is the area shown in this graph:



For example, log 2.5 is the area is between 1 and 2.5.

Lecture 7

Exponential Growth. Euler. Sequences & Series.

Euler's Number e

Euler's number e may be defined in many ways.

The natural logarithm of e is 1:

$$\log e = 1$$

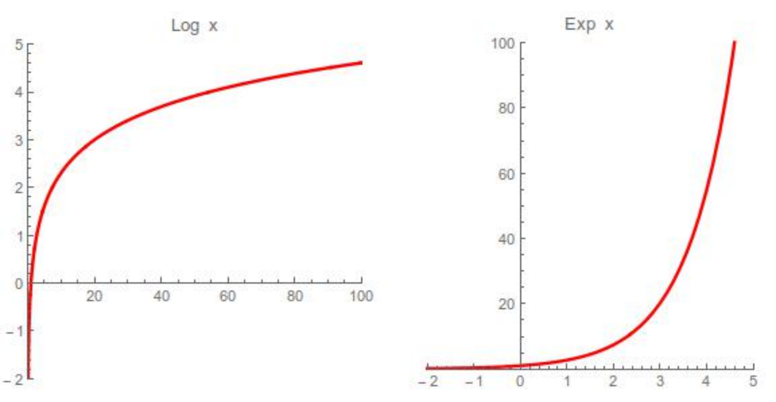
It may be defined as a limit:

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n} \right)^n$$

or as the sum of an infinite series

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

We can get the graph of $\exp x$ from that of $\ln x$ by rotating about the line $x = y$.



Convergence & Divergence of Series

The geometric series converges

$$G = \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 2$$

The harmonic series diverges

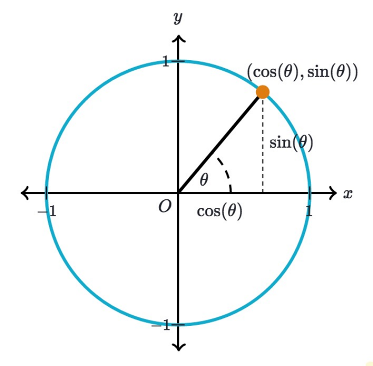
$$H = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \infty$$

For both series, the terms get smaller and smaller and tend towards zero.

Lecture 8

Trigonometry. Taylor Series.

Unit Circle



On the unit circle we have

$$\begin{aligned} x &= \cos(\theta) \\ y &= \sin(\theta) \end{aligned}$$

By Pythagoras' Theorem,

$$x^2 + y^2 = 1$$

Therefore

$$(\cos\theta)^2 + (\sin\theta)^2 = 1$$

Animation of a Sine Wave

Animation showing how the sine function (in red) $y = \sin(\theta)$ is graphed from the y-coordinate (red dot) of a point on the unit circle (in green) at an angle of θ in radians.

`https://en.wikipedia.org/wiki/Sine/`

Lecture 9

Basel Problem. Complex Numbers. Euler's Formula.

Euler showed that

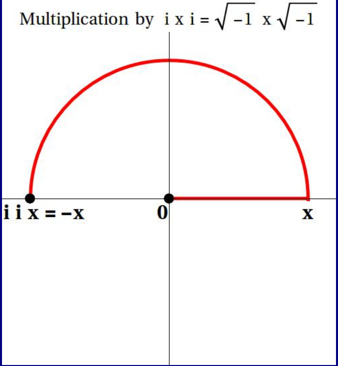
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \right) = \frac{\pi^2}{6}$$

This is our first value of Riemann's ζ-function.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \textbf{so} \qquad \zeta(2) = \frac{\pi^2}{6}$$

We found that when $s = 1$, the series is the divergent harmonic series, so no value of $\zeta(1)$ is defined.

From Number Line to Complex Plane



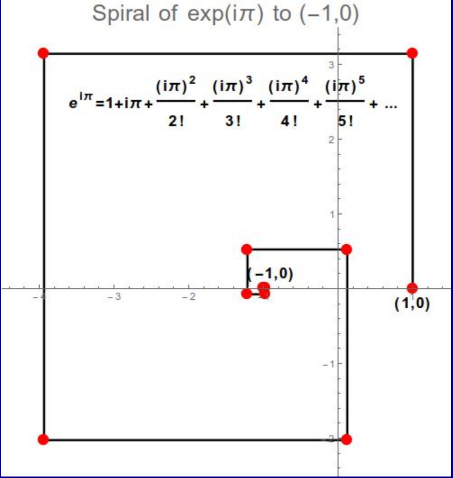
Multiply twice by *i*

This means a rotation of 90° followed by another rotation of 90°.

So operating twice with *i* equals once with -1.

Therefore $i \times i = -1$ which means $i = \sqrt{-1}$.

Spiral of $\exp(i\pi)$ Series to $z = -1$



$$\exp i\pi + 1 = 0$$

Lecture 10

The Prime Number Theorem
and
The Riemann Hypothesis

We will now look at

- Euler's Product Formula
- The Prime Number Theorem
- The Riemann Hypothesis.

Outline

Euler's Product-Sum Formula ($\star$)

Analytic number theory rests on a wonderful formula proved by Leonhard Euler in 1737.

$$\sum_{n=1}^{\infty} n^{-s} = \prod_{\text{primes}} \frac{1}{1 - p^{-s}}$$

We will now look at how he did it.

$\star$: **This section is *"slightly difficult"*.**

By the **Fundamental Theorem of Arithmetic**, any natural number n can be expressed uniquely as

$$n = 2^{e_2} 3^{e_3} 5^{e_5} \cdots p^{e_p}$$

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We raise this to the power $-s$:

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Since every natural number n occurs in the sum, we want a formula that includes every possible combination of prime powers.

Euler found just the right formula.

Adding over all n we get

$$\sum_{n=1}^{\infty} n^{-s} = (1 + 2^{-s} + 2^{-2s} + \cdots) \times (1 + 3^{-s} + 3^{-2s} + \cdots)$$
$$\times (1 + 5^{-s} + 5^{-2s} + \cdots) \cdots$$

Each l.h.s. term n^{-s} arises from a single r.h.s. term.

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Each l.h.s. term n^{-s} arises from a single r.h.s. term.

For each group on the right we have

$$(1 + p^{-s} + p^{-2s} + \cdots) = (1 - p^{-s})^{-1}$$

Adding over all n we get

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Each l.h.s. term n^{-s} arises from a single r.h.s. term.

For each group on the right we have

$$(1 + p^{-s} + p^{-2s} + \cdots) = (1 - p^{-s})^{-1}$$

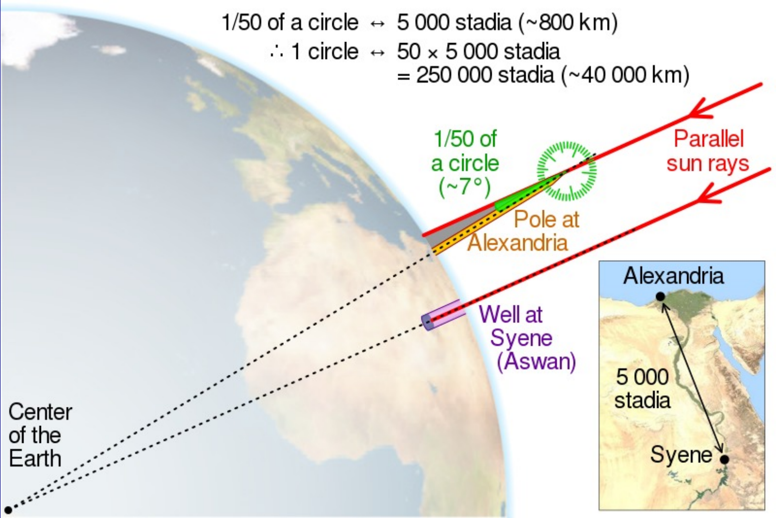
Therefore

$$\sum_{n=1}^{\infty} n^{-s} = \prod_{\text{primes}} (1 - p^{-s})^{-1}$$

This formula relates the Riemann ζ-function to an infinite product over the prime numbers.

Outline

Eratosthenes Measured the Earth



The Sieve of Eratosthenes

Eratosthenes was the Librarian in Alexandria when Archimedes flourished in Syracuse.

They were “pen-pals”.

Eratosthenes estimated size of the Earth.

He devised a systematic procedure for generating the prime numbers: the Sieve of Eratosthenes.

The Sieve of Eratosthenes

The idea:

- List all natural numbers up to n.
- Circle 2 and strike out all multiples of two.
- Move to the next number, 3.
- Circle it and strike out all multiples of 3.
- Continue till no more numbers can be struck out.

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- Continue till no more numbers can be struck out.

The numbers that have been circled are the prime numbers. Nothing else survives.

It is sufficient to go as far as $\sqrt{n}$.

The Sieve of Eratosthenes

	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

The Sieve of Eratosthenes

	2	3		5		7		9	
11		13		15		17		19	
21		23		25		27		29	
31		33		35		37		39	
41		43		45		47		49	
51		53		55		57		59	
61		63		65		67		69	
71		73		75		77		79	
81		83		85		87		89	
91		93		95		97		99	

The Sieve of Eratosthenes

	2	3		5		7			
11		13				17		19	
		23		25				29	
31				35		37			
41		43				47		49	
		53		55				59	
61				65		67			
71		73				77		79	
		83		85				89	
91				95		97			

The Sieve of Eratosthenes

	2	3		5		7			
11		13				17		19	
		23						29	
31						37			
41		43				47		49	
		53						59	
61						67			
71		73				77		79	
		83						89	
91						97			

The Sieve of Eratosthenes

	2	**3**	4	**5**	6	**7**	8	9	10
11	12	**13**	14	15	16	**17**	18	**19**	20
21	22	**23**	24	25	26	27	28	**29**	30
31	32	33	34	35	36	**37**	38	39	40
41	42	**43**	44	45	46	**47**	48	49	50
51	52	**53**	54	55	56	57	58	**59**	60
61	62	63	64	65	66	**67**	68	69	70
71	72	**73**	74	75	76	77	78	**79**	80
81	82	**83**	84	85	86	87	88	**89**	90
91	92	93	94	95	96	**97**	98	99	100

Is There a Pattern in the Primes?

It is a simple matter to make a list of
all the primes less that 100 or 1000.

It becomes clear very soon that there is
no clear pattern emerging.

The primes appear to be scattered at random.

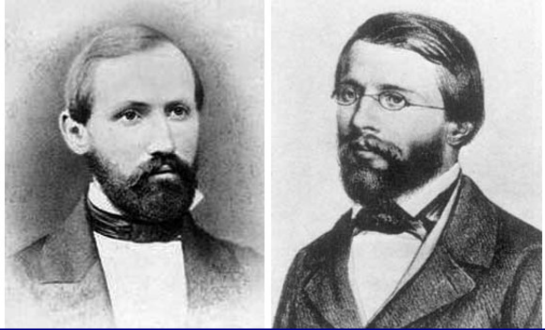
Figure : Prime numbers up to 100

The grand challenge is to find patterns in the sequence of prime numbers.

This is an enormously difficult problem that has taxed the imagination of the greatest mathematicians for centuries.

Outline

Bernhard Riemann (1826–1856)



Bernhard Riemann (1826-66)

Bernhard Riemann (1826–1866)

- Born in Breselenz in Hanover.
- Son of a Lutheran pastor.
- Timid and reserved by nature.
- School in Luneburg: teacher noticed his talents.
- Mastered Legendre's Theory of Numbers.

Bernhard Riemann (1826–1866)

In 1846 Riemann began his studies at Göttingen (Gauss still working but near the end of his career).

Moved to Berlin, where Dirichlet worked.

1851: Riemann awarded a doctorate (in Göttingen).

Thesis on the foundations of complex variable theory.

Bernhard Riemann (1826–1866)

1853: Riemann presented work on trigonometric series for his Habilitation.

Constructed what we now call the Riemann integral.

The Foundations of Geometry: Reimann's vision of geometry was profound in its sweeping generality,

Riemannian Geometry was the framework for Einstein's General Theory of Relativity.

Bernhard Riemann (1826–1866)

In 1859, Riemann appointed Professor at Göttingen.

Elected member of Berlin Academy (also 1859).

Presented his single contribution to number theory, on the distribution of prime numbers:

Contained the conjecture: Riemann's Hypothesis.

Some of Riemann's Mathematics

- Complex variable theory
- Number theory
- Geometry
- Integration
- Calculus of variations
- Theory of electricity.

Some of Riemann's Mathematics

- Complex variable theory
- Number theory
- Geometry
- Integration
- Calculus of variations
- Theory of electricity.

- Riemann integral
- Riemann mapping theorem
- Riemann sphere
- Riemann sheets
- Riemann curvature tensor
- Cauchy-Riemann equations
- Riemannian manifolds.

Outline

A Fragment of History

In 1792 Carl Friedrich Gauss, then only 15 years old, found that the proportion of primes less that n decreased approximately as $1/\log n$.

Around 1795 Legendre noticed a similar logarithmic pattern of the primes, but it was to take another century before a proof emerged.

In a letter dated 1823, the Norwegian mathematician Niels Henrik Abel described the distribution of primes as "the most remarkable result in all of mathematics."

In 1838, Dirichlet discovered an approximation to $\pi(n)$ using the logarithmic integral

$$\pi(n) \approx \mathrm{Li}(n) = \int_2^n \frac{\mathrm{d}x}{\log x}$$

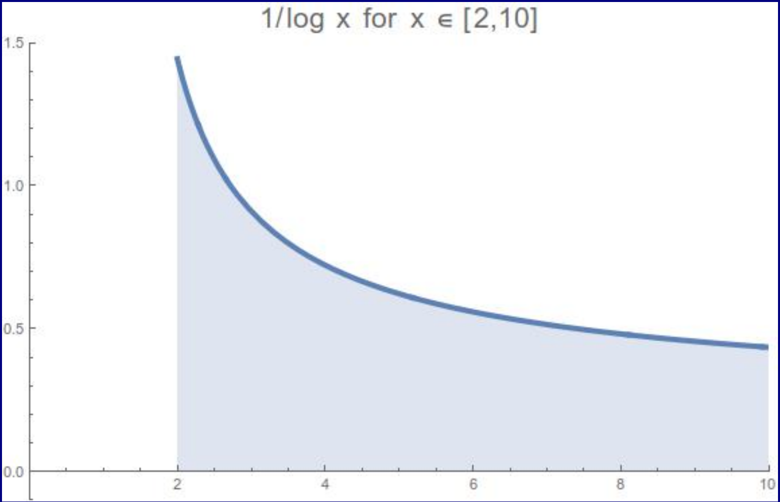
This gives a significantly better estimate of $\pi(n)$ than the simple ratio $n/\log n$.

In 1838, Dirichlet discovered an approximation to $\pi(n)$ using the logarithmic integral

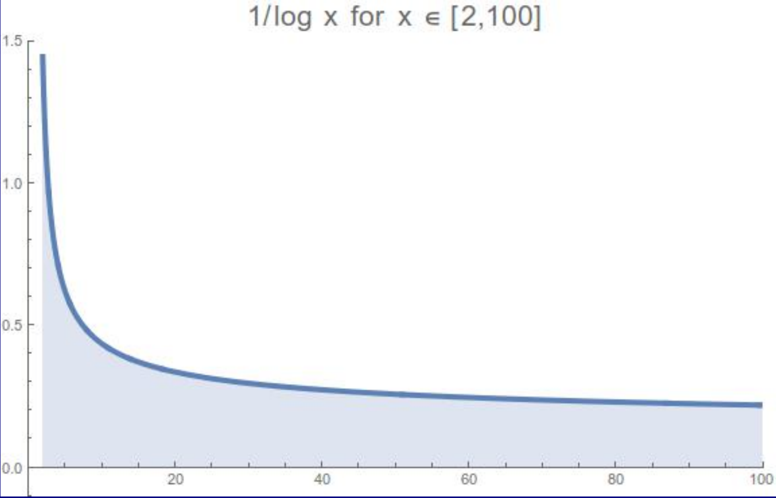
$$\pi(n) \approx \mathrm{Li}(n) = \int_2^n \frac{\mathrm{d}x}{\log x}$$

This gives a significantly better estimate of $\pi(n)$ than the simple ratio $n/\log n$.

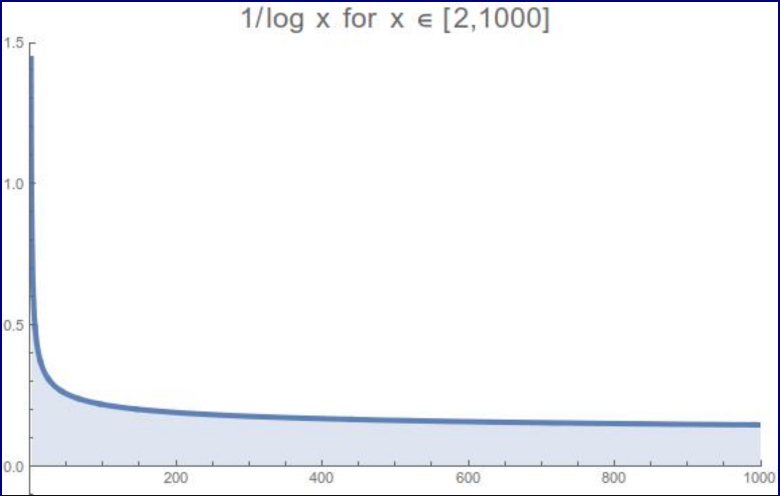
The integral, $\int$, is just the area under the graph of the function $y(x) = 1/\log x$.



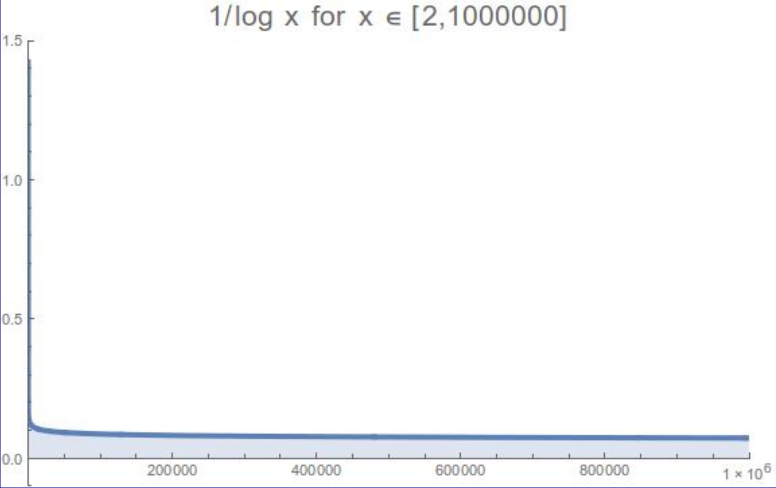
1/log x for x ∈ [2,10]
1.5
1.0
0.5
0.0
2
4
6
8
10

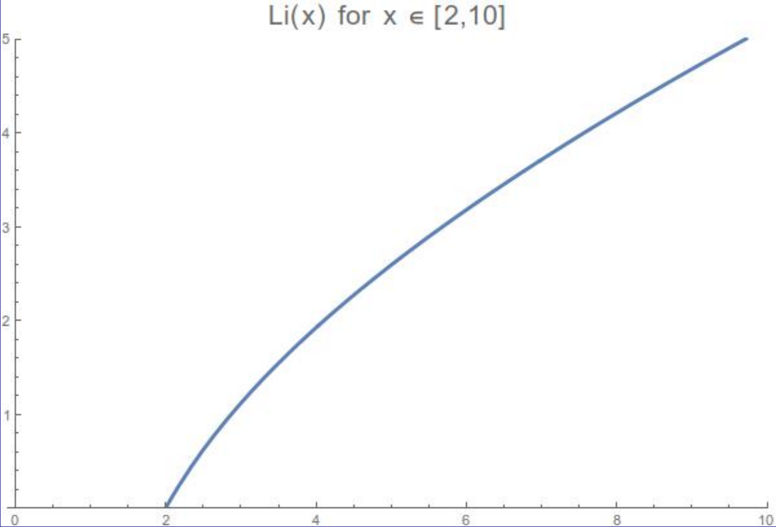


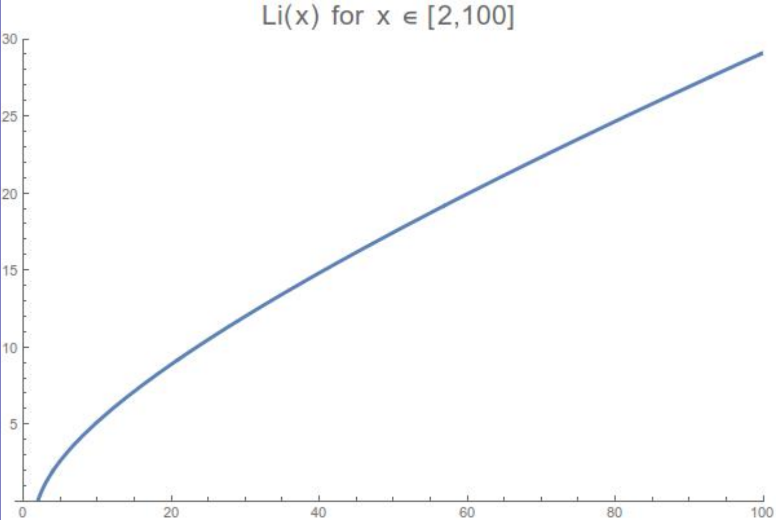
1/log x for x ∈ [2,100]
1.5
1.0
0.5
0.0
20
40
60
80
100

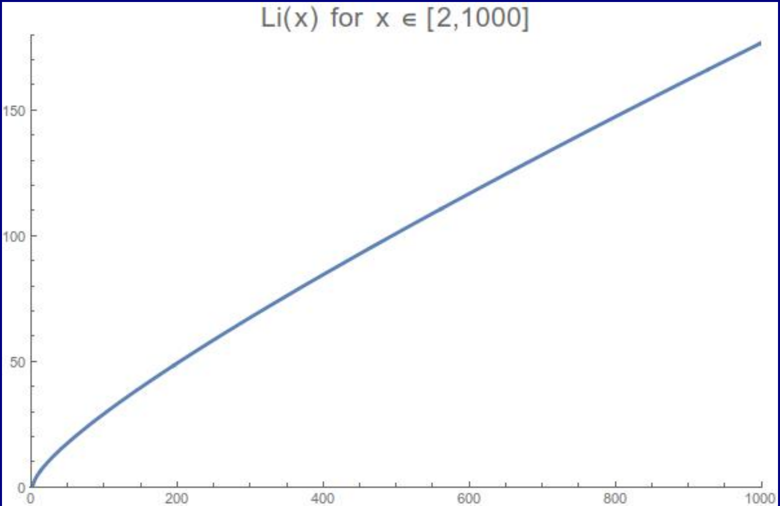


1/log x for x ∈ [2,1000]
1.5
1.0
0.5
0.0
200
400
600
800
1000

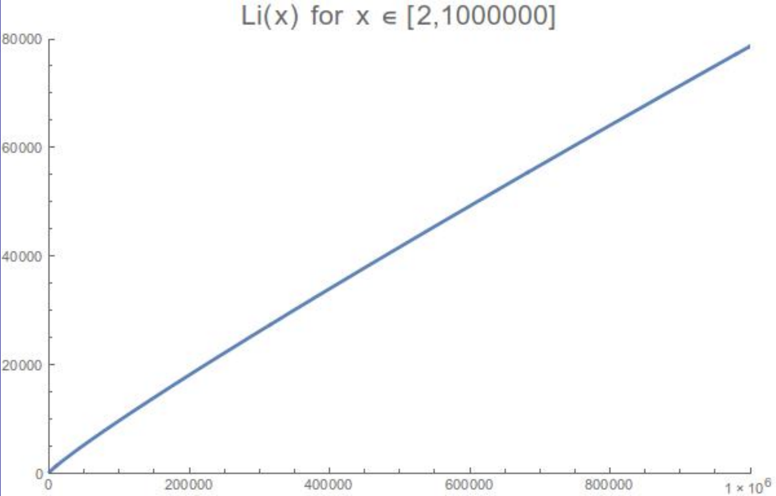

1/log x for x ∈ [2,1000000]
1.5
1.0
0.5
0.0
200000
400000
600000
800000
1 × 10^6


Li(x) for x ∈ [2,10]
5
4
3
2
1
0
2
4
6
8
10


Li(x) for x ∈ [2,100]
30
25
20
15
10
5
0
20
40
60
80
100



Li(x) for x ∈ [2,1000]
150
100
50
0
0
200
400
600
800
1000


Li(x) for x ∈ [2,1000000]
80000
60000
40000
20000
0
0
200000
400000
600000
800000
1 × 10^6

Die Anzahl der Primzahlen . . .

In 1859, Bernhard Riemann published a paper on the distribution of the prime numbers.

This was his only publication on this topic and was, like all his other contributions to mathematics, a result of singular importance.

Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.

Bernhard Riemann

[Monatsberichte der Berliner Akademie, November 1859.]

Transcribed by D. R. Wilkins

Preliminary Version: December 1998

Riemann's major discovery was the link between the primes and the zeros of a complex function now called the Riemann zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

This is the main reason why the zeta function has such importance in number theory.

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This is the main reason why the zeta function has such importance in number theory.

Euler had earlier discovered the connection between the zeta function and prime numbers:

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$$

Riemann's work inspired two proofs of the PNT, discovered independently by Jacques Hadamard and Charles de la Vallée Poussin, both in 1896.

The Prime Number Theorem

God may not play dice with the Universe, but something strange is going on with the prime numbers (Paul Erdős, after Albert Einstein)

Primes are the atoms of the number system . . .
and have fascinated mathematicians for millennia.

Euclid gave a remarkably simple proof
that that there is an infinity of primes.

Many questions about primes remain to be answered.

Prime Pairs

A simple example is the set of prime pairs, $(p, p+2)$. All the evidence suggests that this set is infinite, but this has never been proved.

Primes occur with decreasing frequency as their magnitude grows.

Since there are more potential divisors with increasing magnitude, it is unsurprising that the density of primes becomes less.

Percentage of Primes Less than N

Table : Percentage of Primes less than N

100	25	25.0%
1,000	168	16.8%
1,000,000	78,498	7.8%
1,000,000,000	50,847,534	5.1%
1,000,000,000,000	37,607,912,018	3.8%

We can see that the percentage of primes is falling off with increasing size.

But the rate of decrease is very slow.

The Prime Counting Function

We define the prime counting function $\pi(n)$:
$\pi(n)$ is the number of primes $\leq n$.

$$
\begin{aligned}
\pi(1) &= 0 \\
\pi(2) &= 1 \\
\pi(3) = \pi(4) &= 2 \\
\pi(5) = \pi(6) &= 3 \\
\pi(7) = \pi(8) = \pi(9) = \pi(10) &= 4 \\
\pi(11) = \pi(12) &= 5 \\
&\cdots \\
\pi(100) &= 25
\end{aligned}
$$

Prime Staircase Graph

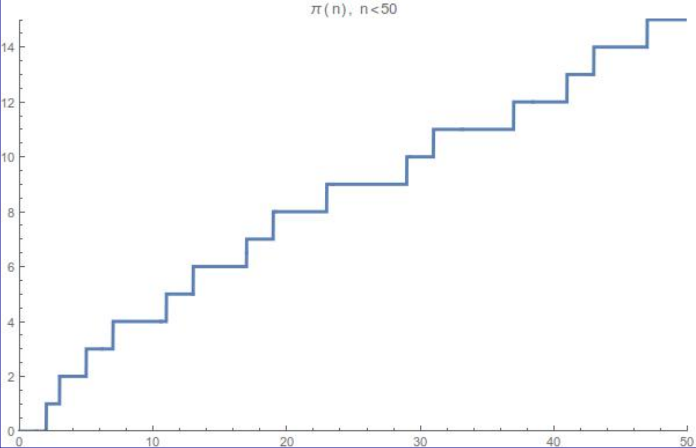


Figure : The prime counting function $\pi(n)$ for $0 \leq n \leq 50$.

The Prime Number Theorem (PNT)

The Prime Number Theorem describes the asymptotic distribution of the prime numbers.

The proportion of primes less than n is:

$$\pi(n)/n.$$

The PNT states that this is asymptotic to $1/\log n$

$$\lim_{n\to\infty}\left[\frac{\pi(n)}{n/\log n}\right] = 1\,.$$

Again, PNT states that

$$\lim_{n\to\infty}\left[\frac{\pi(n)}{n/\log n}\right] = 1\ .$$

We also write

$$\pi(n) \sim \frac{n}{\log n}$$

PNT is equivalent to saying that the n-th prime number p_n is given approximately by $p_n \approx n\log n$.

PNT $$\pi(n) \sim \frac{n}{\log n}$$

The relative error approaches zero as n increases.

However, the primes are not distributed evenly, as suggested by this formula, but have a substantial element of randomness.

The pattern of primes is one of the most intriguing issues in all of mathematics.

Prime Staircase Graph to 50

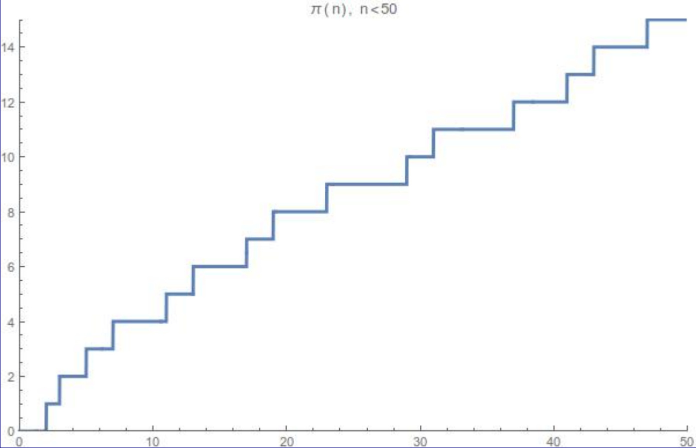


Figure : The prime counting function $\pi(n)$ for $0 \leq n \leq 50$.

Prime Staircase Graph to 500

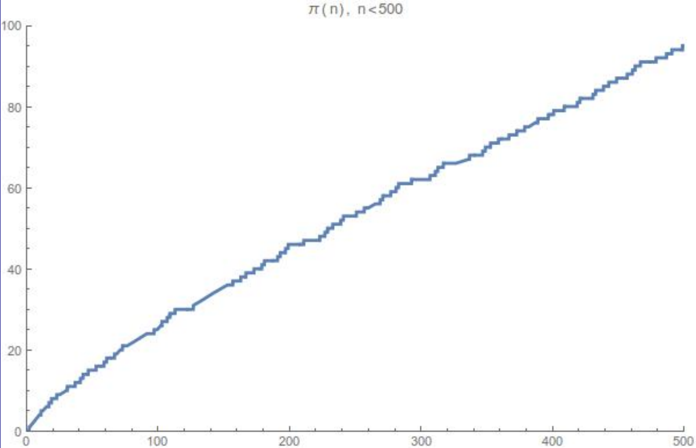


Figure : The prime counting function $\pi(n)$ for $0 \leq n \leq 500$.

Prime Staircase Graph to 5000

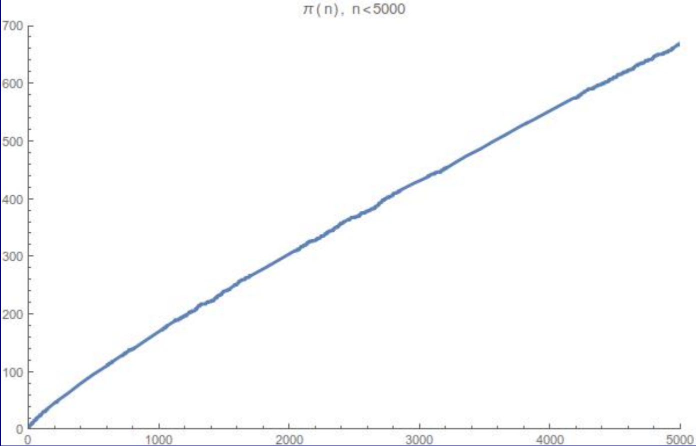


Figure : The prime counting function $\pi(n)$ for $0 \leq n \leq 5000$.

Prime Staircase Graph to 50000

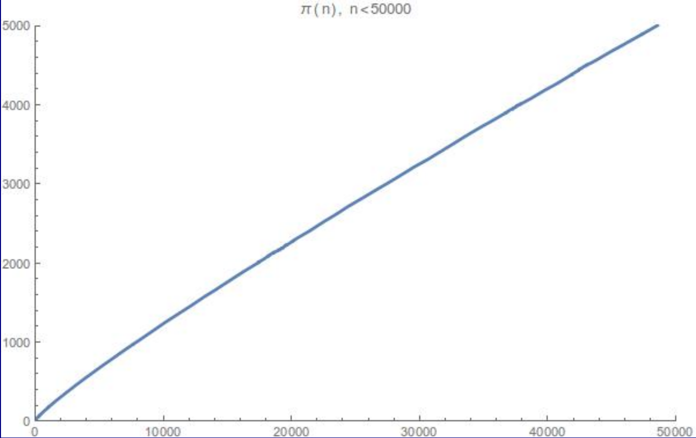
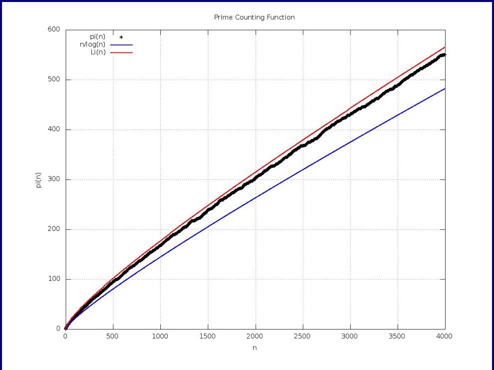
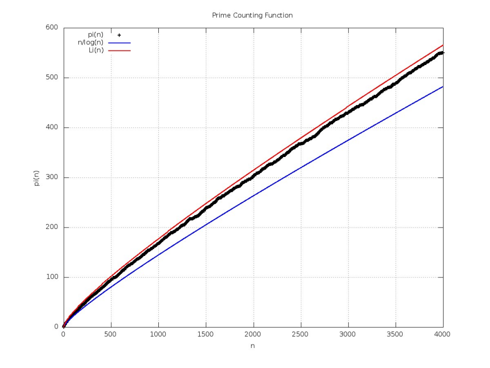


Figure : The prime counting function $\pi(n)$ for $0 \leq n \leq 50000$.

The function $\pi(n)$ is shown by the heavy black curve. There is a discontinuity at each prime number.





The blue curve is the graph of $n/\log n$, which gives a reasonable approximation to $\pi(n)$. While the ratio tends to 1, the difference increases with n.

The upper curve (red) is the logarithmic integral $\mathrm{Li}\, n$, which yields a significantly better estimate of $\pi(n)$.

Outline

The Riemann Hypothesis

The most important unsolved problem in mathematics.

Posed more than 150 years ago by Bernhard Riemann.

Problem concerns the zeros of a mathematical function called the Riemann zeta function.

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Problem concerns the zeros of a mathematical function called the Riemann zeta function.

One of 23 problems listed by David Hilbert in 1900.

One of seven Millennium Problems: a million dollar prize on offer from The Clay Mathematics Institute.

Riemann hypothesis intimately related to the distribution of the prime numbers.

The Riemann Hypothesis

Riemann's conjecture was this:

All the zeros of the zeta function with positive real parts are on a single vertical line in the *s*-plane.

$$\Re\{s\} \geq 0 \text{ and } \zeta(s) = 0 \quad \Longrightarrow \quad \Re\{s\} = \tfrac{1}{2}$$

The Riemann Hypothesis

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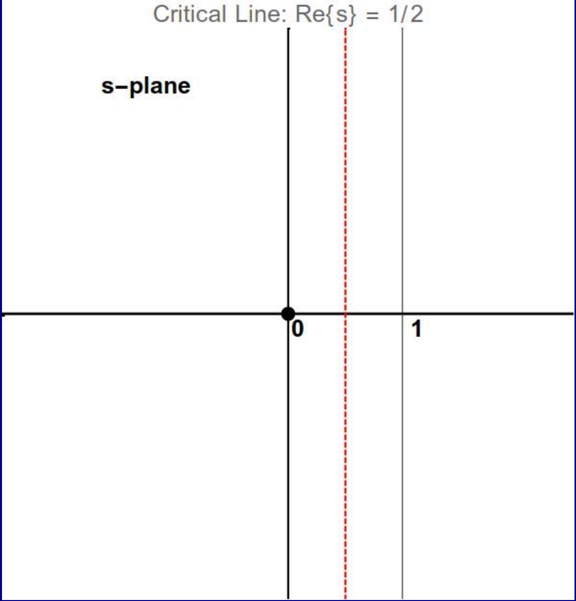
All the zeros of the zeta function with positive real parts are on a single vertical line in the s-plane.

$$\Re\{s\} \geq 0 \text{ and } \zeta(s) = 0 \quad \Longrightarrow \quad \Re\{s\} = \tfrac{1}{2}$$

It is known that there are zeros at the negative even integers, $\{-2, -4, -6 \ \ldots\}$. These are the "trivial zeros".

The nontrivial zeros of the Riemann zeta function all lie on the critical line $\Re(s) = \frac{1}{2}$.


Critical Line: Re{s} = 1/2
s–plane
0
1

Equivalent Statement

Colloquially, an integer has an equal chance of having an odd or an even number of prime factors.

$$26 = 2 \times 13 \qquad 27 = 3 \times 3 \times 3$$

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We will make this statement precise.

Let $\omega(n)$ be the number of prime factors of n, counted with multiplicity.

$$\omega(26) = 2 \qquad \omega(27) = 3$$

The Liouville function is defined as

$$\lambda(n) = (-1)^{\omega(n)}$$

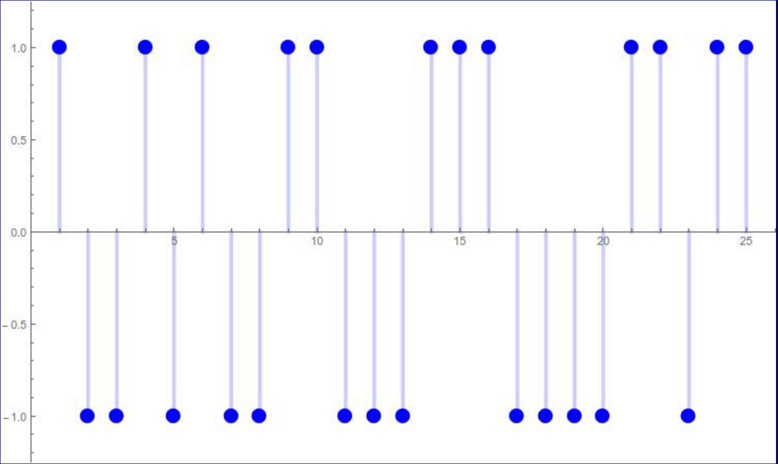
In his doctoral thesis of 1899, Edmund Landau proved this Theorem:

RH is equivalent to

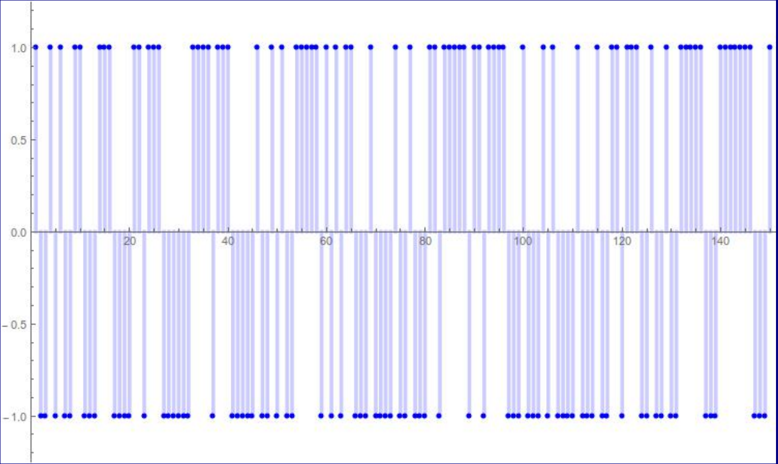
$$\lim_{n\to\infty}\left[\frac{\lambda(1)+\lambda(2)+\cdots+\lambda(n)}{n^{\frac{1}{2}+\epsilon}}\right] = 0\,.$$

for all $\epsilon > 0$.

Liouville Function $\lambda(n)$ for $n \leq 25$



Liouville Function $\lambda(n)$ for $n \leq 150$



Extending the Zeta-function

Something is fishy!

The Riemann Hypothesis says something about zeros with real part equal to $\frac{1}{2}$.

But the infinite series defining $\zeta(s)$ **DOES NOT CONVERGE FOR** $\Re\{s\} \leq 1$.

So, we have to find a way to extend the function.

This process is called **analytic continuation.**

The Riemann Zeta-function

The "zeta function" is defined by an infinite series:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

This series converges for $\Re\{s\} > 1$.

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We can extend $\zeta(s)$ to the whole right-hand plane.

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This series converges for $\Re\{s\} > 1$.

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$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$$

is the "eta-function", which converges for $\Re\{s\} > 0$.

Recall that the harmonic series diverges, while the alternating harmonic series converges (to $\log 2$).

It is a fairly simple exercise to show that

$$\zeta(s) = \frac{\eta(s)}{1 - 2^{1-s}}$$

This allows us to evaluate $\zeta(s)$ for all s with $\Re\{s\} > 0$.

This process is called analytic continuation.

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$$\zeta(s) = \frac{\eta(s)}{1 - 2^{1-s}}$$

This allows us to evaluate $\zeta(s)$ for all s with $\Re\{s\} > 0$.

This process is called analytic continuation.

We can actually extend $\zeta(s)$ to the whole complex plane except for $s = 1$, where there is an infinity.

Riemann found a way to extend the zeta function to the entire complex plane except for the pole at $s = 1$. That is, to the set $\mathbb{C} - \{1\}$.

Zeros on the Critical Line

- **Hardy** (1914) showed that $\zeta(s)$ has infinitely many zeros on the critical line.
- **Selberg** (1942) showed that a positive proportion of zeros lie on the line.
- **Levinson** (1974) showed that at least $\frac{1}{3}$ of the zeros are on the line.
- **Conrey** (1989) showed that at least $\frac{2}{5}$ of the zeros are on the line.

Riemann Spectrum to Prime Numbers

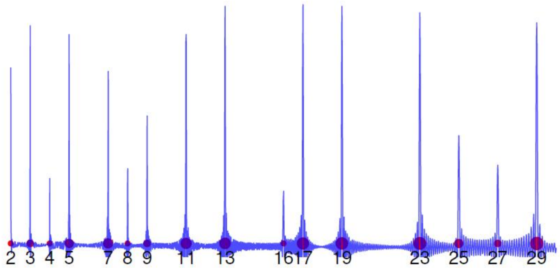


Figure 35.1: Illustration of $-\sum_{i=1}^{1000} \cos(\log(s)\theta_i)$, where $\theta_1 \sim 14.13, \ldots$ are the first 1000 contributions to the Riemann spectrum. The red dots are at the prime powers p^n, whose size is proportional to $\log(p)$.

Riemann Spectrum to Prime Numbers

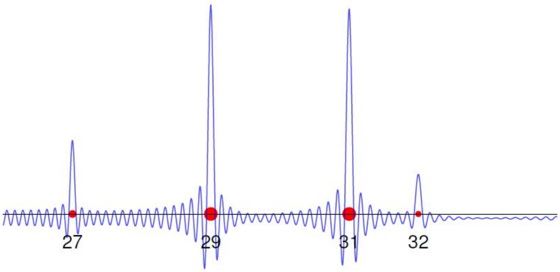


Figure 35.2: Illustration of $-\sum_{i=1}^{1000} \cos(\log(s)\theta_i)$ in the neighborhood of a twin prime. Notice how the two primes 29 and 31 are separated out by the Fourier series, and how the prime powers 3^3 and 2^5 also appear.

Riemann Spectrum to Prime Numbers

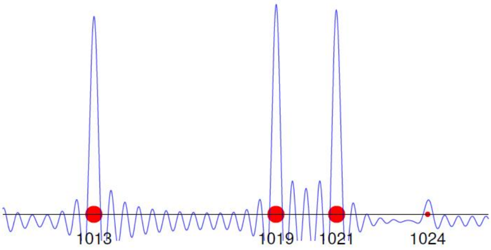
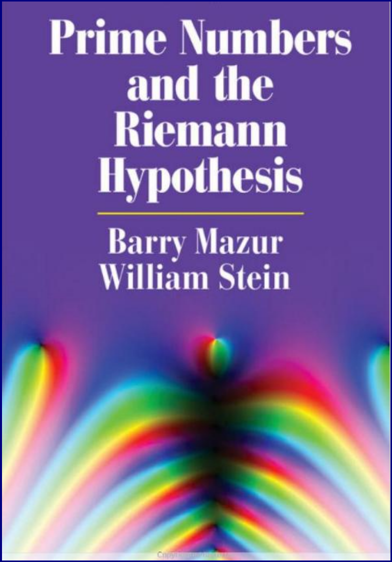
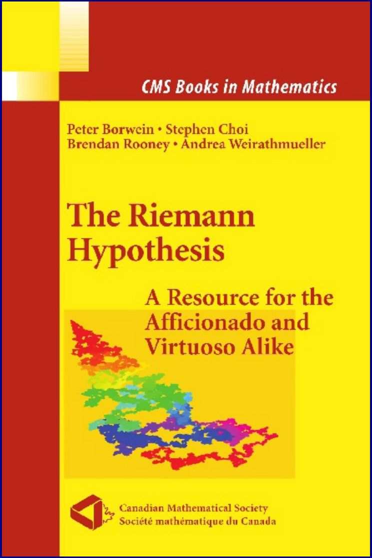


Figure 35.3: Fourier series from 1, 000 to 1, 030 using 15,000 of the numbers θ_i. Note the twin primes 1,019 and 1,021 and that $1{,}024 = 2^{10}$.


Prime Numbers and the Riemann Hypothesis
Barry Mazur
William Stein


CMS Books in Mathematics
Peter Borwein • Stephen Choi
Brendan Rooney • Andrea Weirathmueller
The Riemann Hypothesis
A Resource for the Afficionado and Virtuoso Alike
Canadian Mathematical Society
Société mathématique du Canada

True or False?

There is powerful heuristic evidence for RH:
it has been checked that the first ten trillion
zeros of the zeta-function lie on the critical line.

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To a non-mathematician, this amounts to proof.

True or False?

There is powerful heuristic evidence for RH: it has been checked that the first ten trillion zeros of the zeta-function lie on the critical line.

To a non-mathematician, this amounts to proof.

On the other hand, there are examples of mathematical hypotheses that hold true for any computationally accessible range, but that are known to be false.

Nobody knows. Best of luck proving the RH. I hope you win the million dollar prize.

Thank you