AweSums:

The Majesty of Mathematics

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Outline

Outline

AweSums: The Majesty of Maths

Bernhard Riemann (1826-66)

AweSums: The Majesty of Maths

We aim to get a flavour of the Riemann Hypothesis.

It involves the zeros of the "Zeta function":

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

So, we need to talk about several *new topics.*

In this lecture, we will look at complex numbers.

Outline

Polynomials

Many functions can be approximated by polynomials.

What is a polynomial? A simple algebraic function, a combination of integral powers of the variable x.

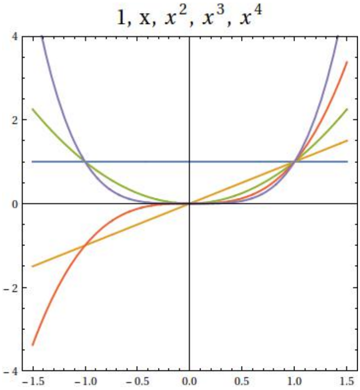
Examples of polynomials:

Linear:	$5x - 7$
Quadratic:	$x^2 + 3x + 4$
Cubic:	$x^3 + 3x^2 + 4x - 5$
***n*-th order:**	$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$
Cubic with roots:	$6(x-3)(x-5)(x+2)$

Describe Polynomials on BB

Outline the properties and graphs of simple polynomials on the blackboard.

Basis Functions for Approximation



Many functions can be approximated by a series of polynomial functions.

Here we plot the functions

$$1 \quad x \quad x^2 \quad x^3 \quad x^4$$

which are used as basis functions.

Polynomial Approximation. Taylor Series

Any "reasonable function" $f(x)$ can usually be approximated by a simple polynomial function

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

Sometimes we can find the roots of the polynomial; that is, the values of x for which it is zero.

Then we are able to write the polynomial as

$$p(x) = a_n(x - x_1)(x - x_2)(x - x_3) \cdots (x - x_n)$$

It is simple to sketch the graph of this function.

Real and Complex Roots

Explain on blackboard, with graphs, how the roots of polynomials of various degrees appear.

A quadratic may have two distinct roots, a single (repeated) root, or no (real) root at all.

Taylor Series for Sine Wave

The Taylor series for $\sin x$ **is**

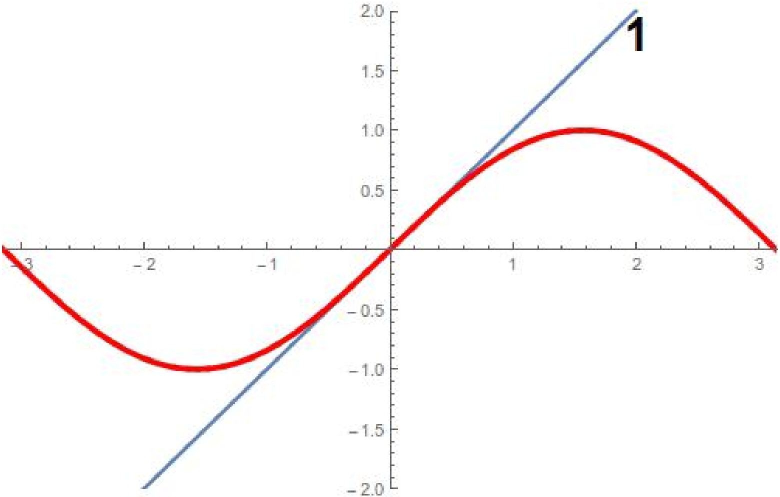
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

We truncate to get a sequence of polynomials:

$$\begin{aligned} p_1(x) &= x \\ p_3(x) &= x - \tfrac{x^3}{3!} \\ p_5(x) &= x - \tfrac{x^3}{3!} + \tfrac{x^5}{5!} \\ p_7(x) &= x - \tfrac{x^3}{3!} + \tfrac{x^5}{5!} - \tfrac{x^7}{7!} \end{aligned}$$

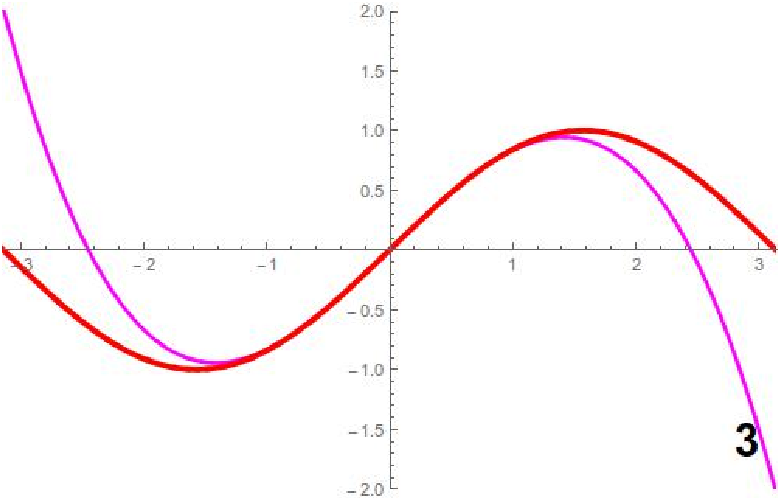
They approximate $\sin x$ **better with increasing order.**

Polynomial Approximation to Sine Wave



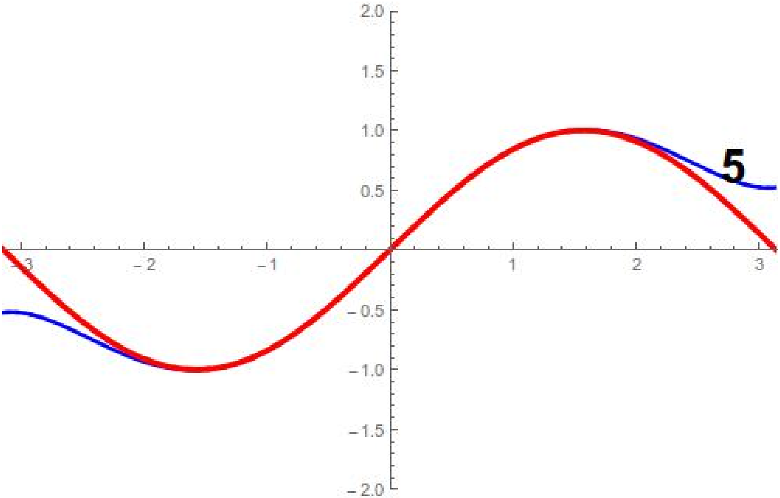
$p_1(x)$ **is a good fit only near** $x = 0$**.**

Polynomial Approximation to Sine Wave



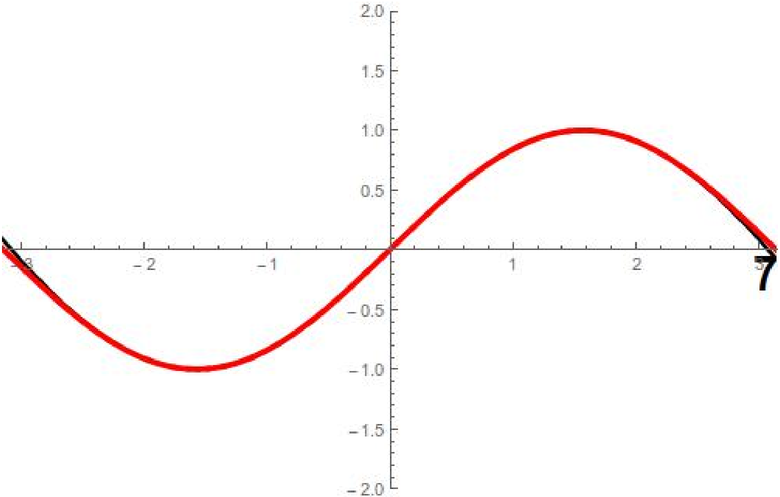
$p_3(x)$ **fits to about** $x = \pi/3$**.**

Polynomial Approximation to Sine Wave



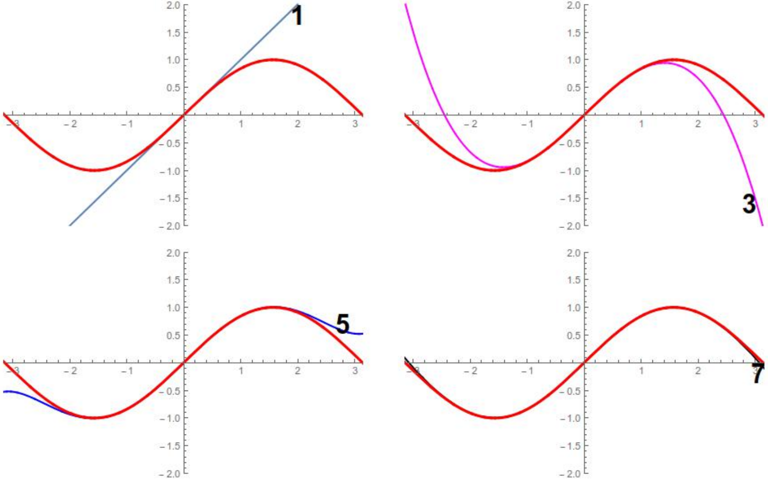
$p_5(x)$ **gets "over the first hump".**

Polynomial Approximation to Sine Wave



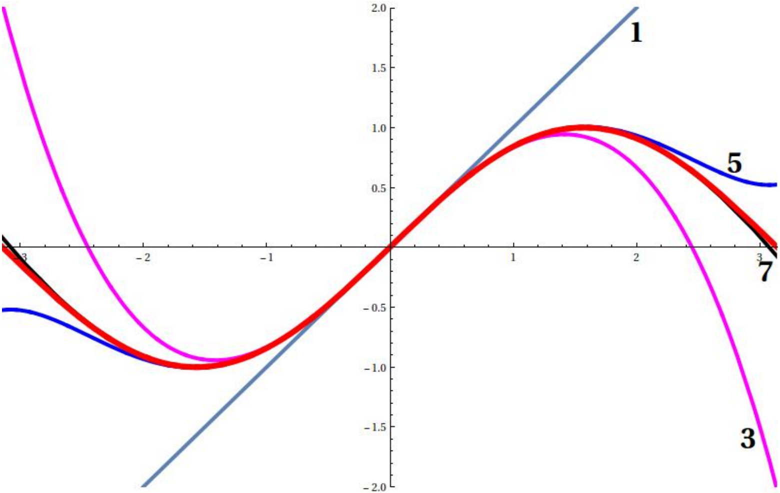
$p_7(x)$ **is a good fit over a full wavelength.**

Polynomial Approximation to Sine Wave



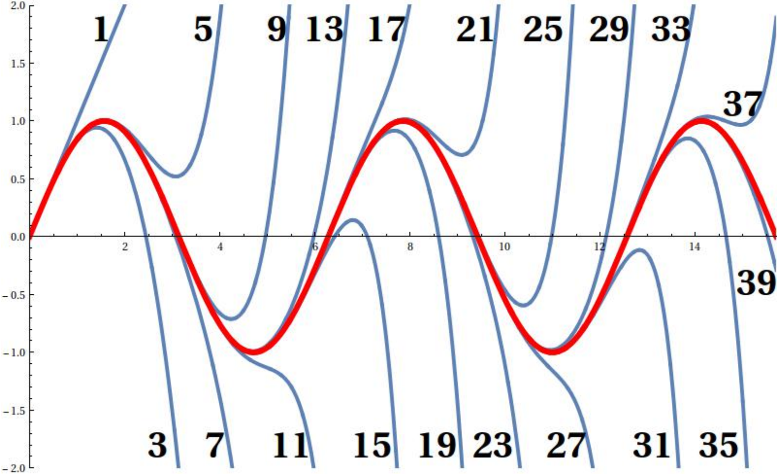
First four approximations over a full wavelength.

Polynomial Approximation to Sine Wave



First four approximations over a full wavelength.

Polynomial Approximation to Sine Wave



$p_{39}(x)$ **fits well over five wavelengths.**

Taylor Series for Cosine Wave

The Taylor series for $\cos x$ **is**

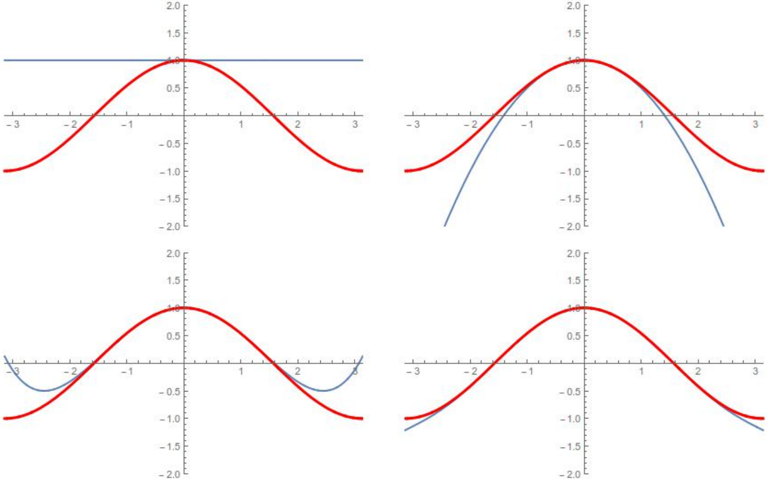
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

We truncate to get a sequence of polynomials:

$$\begin{aligned} p_0(x) &= 1 \\ p_2(x) &= 1 - \tfrac{x^2}{2!} \\ p_4(x) &= 1 - \tfrac{x^2}{2!} + \tfrac{x^4}{4!} \\ p_6(x) &= 1 - \tfrac{x^2}{2!} + \tfrac{x^4}{4!} - \tfrac{x^6}{6!} \end{aligned}$$

They approximate $\cos x$ **better with increasing order.**

Approximations to Cosine Wave



Four approximations to $\cos x$**.**

The Exponential Function

We have defined the exponential function as

- The inverse of the logarithmic function
- The limit of the sequence s_n where

$$s_n = \left(1 + \frac{x}{n}\right)^n$$

as $n \longrightarrow \infty$

Now we will define it by an infinite series.

The Binomial Expansion

If you have never heard of the binomial theorem please ignore this slide

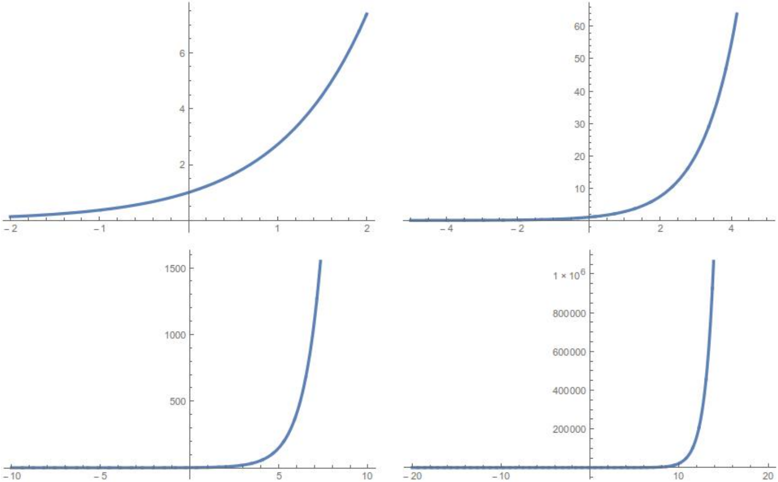
If $s_n = (1 + x/n)^n$ is expanded using the binomial theorem, we get the following expression:

$$s_n = 1 + n\left(\frac{x}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{x^2}{n^2}\right) + \frac{n(n-1)(n-2)}{3!}\left(\frac{x^3}{n^3}\right) + \cdots + \left(\frac{x^n}{n!}\right)$$

Letting n become large, this tends to the series

$$s_n \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

The Exponential Function



Exponential function on four ranges.

Taylor Series for Exponential Function

The Taylor series for $\exp x$ **is**

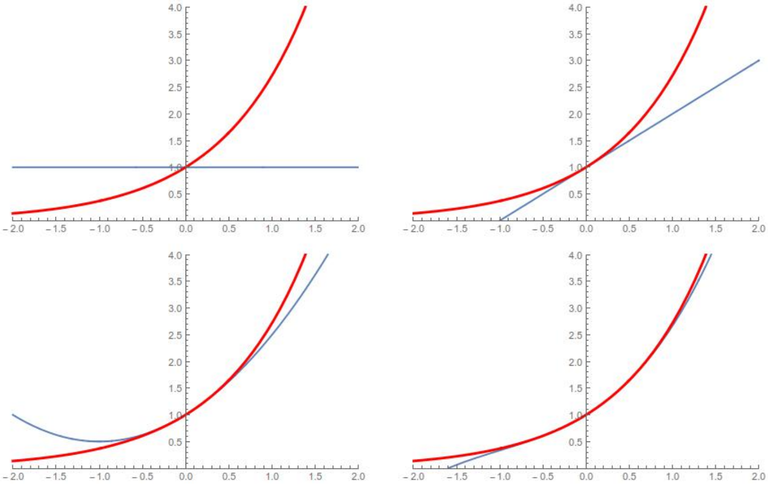
$$\exp x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

We truncate to get a sequence of polynomials:

$$\begin{aligned} p_0(x) &= 1 \\ p_1(x) &= 1 + x \\ p_2(x) &= 1 + x + \frac{x^2}{2!} \\ p_3(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \end{aligned}$$

They approximate $\exp x$ **better with increasing order.**

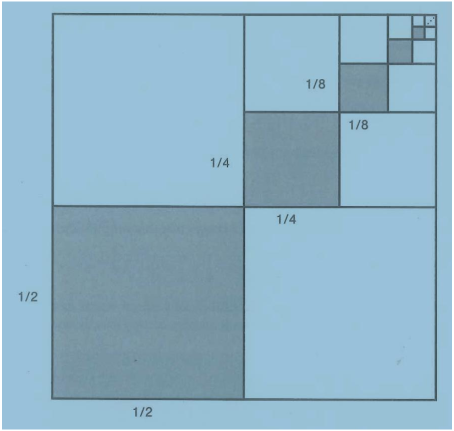
Approximations to Exponential Function



Four approximations to $\exp x$**.**

Outline

Distraction 8: Sum by Inspection



We will find the shaded area without calculation

Proof by Inspection

Look at the figure in two different ways

At each scale, we have three squares the same size, and we keep one of them (black) and omit the others.

So, the area of the shaded squares is $\frac{1}{3}$.

However, it is also given by the series

$$\left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{16}\right)^2 + \cdots$$

Therefore we can sum the series:

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \cdots \quad = \quad \frac{1}{3}$$

Outline

The Basel Problem

Many mathematicians tried and failed to find the sum of the series of inverse squares:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \right)$$

Leibniz and the Bernoullis were unsuccessful.

In 1734 Leonhard Euler found the sum by a virtuoso performance.

We will now look at how he did it.

Taylor Series for Sine Wave

The Taylor series for $\sin x$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

The coefficient of x^3 is $-\frac{1}{6}$.

We will express $\sin x$ in another way and find an alternative expression for the coeficient of x^3.

Equating the two expressions will give a solution of the Basel Problem.

The zeros of the function $\sin x$ **are at the points**

$$\cdots \quad -3\pi \quad -2\pi \quad -\pi \quad 0 \quad \pi \quad 2\pi \quad 3\pi \quad \cdots$$

Euler expressed $\sin x$ **in terms of the roots:**

$$\sin x = B\,x\left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\cdots$$

where B **is a constant (**$\lim_{x\to 0}\frac{\sin x}{x} = 1$ **implies** $B = 1$**).**

We can alternatively write this as

$$\sin x = Bx\left[1-\left(\frac{x}{\pi}\right)^2\right]\left[1-\left(\frac{x}{2\pi}\right)^2\right]\left[1-\left(\frac{x}{3\pi}\right)^2\right]\cdots$$

$$\sin x = x\left[1-\left(\frac{x}{\pi}\right)^2\right]\left[1-\left(\frac{x}{2\pi}\right)^2\right]\left[1-\left(\frac{x}{3\pi}\right)^2\right]\cdots$$

Multiplying out, the coefficient of x^3 is

$$-\left(\frac{1}{\pi}\right)^2-\left(\frac{1}{2\pi}\right)^2-\left(\frac{1}{3\pi}\right)^2-\cdots$$

But this must equate to the coefficient $-\frac{1}{6}$ from the Taylor series:

$$-\left(\frac{1}{\pi}\right)^2-\left(\frac{1}{2\pi}\right)^2-\left(\frac{1}{3\pi}\right)^2-\cdots=-\frac{1}{6}$$

Therefore

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right)=\frac{\pi^2}{6}$$

Euler's bravura solution of the Basel Problem is:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \right) = \frac{\pi^2}{6}$$

The rate of convergence is surprisingly slow:
One million terms give only six digits of accuracy.

Table : Convergence of Basel Problem Series

10 terms	**Sum = 1.549768**
100 terms	**Sum = 1.634984**
1 000 terms	**Sum = 1.643935**
10 000 terms	**Sum = 1.644834**
100 000 terms	**Sum = 1.644924**
1 000 000 terms	**Sum = 1.644933**
$\pi^2/6$	**Sum = 1.644934**

The result is

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \right) = \frac{\pi^2}{6}$$

This is our first value of Riemann's ζ-function.

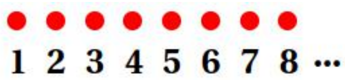
$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \text{so} \qquad \zeta(2) = \frac{\pi^2}{6}$$

We found that when $s = 1$, the series is the divergent *harmonic series,* so no value of $\zeta(1)$ is defined.

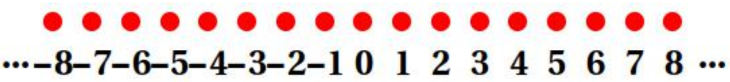
Outline

Positive and Negative Integers

The natural numbers arose at an early stage:



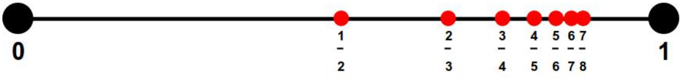
Around 1550, negative numbers came into use.



This suggests that the concept of negative numbers was difficult for mathematicians to grasp.

The Real Number Line

Gradually, new types of number were recognised. The gaps in the number line were "filled in".



Still, all the numbers could be arranged on a line.

In the fifteenth century, the numbers "broke out" and spread all over the plane.

Extending Numbers to Solve Equations

A simple linear equation, $ax = b$, with a and b positive, is easy to solve: just divide by a.

However, an equation like $ax + b = 0$ ($a > 0$ and $b > 0$) can be solved only if negative quantities are admitted.

Mathematicians of the *Italian Renaissance* were the first to solve equations with negative quantities.

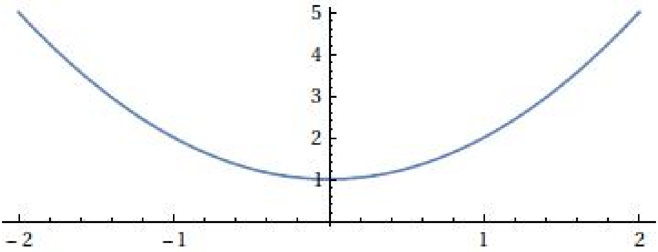
Del Ferro, Tartaglia, Cardano, Ferrari and Bombelli were foremost amongst these.

They found solutions to cubic and quartic equations.

The Need for New Numbers

Quadratic equations like $ax^2 + bx + c = 0$ could be solved in some circumstances.

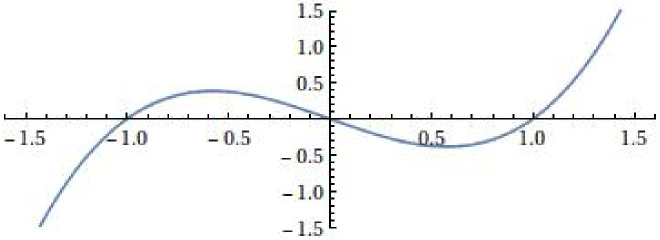
In other cases, there was no solution:



It is clear that $y = x^2 + 1$ does not vanish on the x-axis.

The Need for New Numbers

Cubic equations like $ax^3 + bx^2 + cx + d = 0$ always have a root: the graph always crosses the x-axis.



But the Cardano formula for the solution sometimes involves square roots of negative quantities.

This forced mathematicians to consider "imaginary" quantities.

Numbers as Operators

We can interpret the product of two numbers

$$a \times b$$

as the number a **operating on the number** b**.**

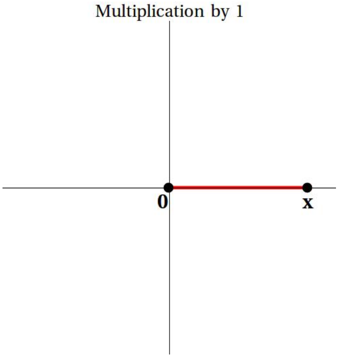
For example, $2 \times b$ **corresponds to the operation of doubling the number** b**.**

It is remarkable that "a **operating on** b**" gives the same result as "**b **operating on** a**"**

$$a \times b = b \times a$$

We see that a **and** b **are both operators and numbers.**

From Number Line to Complex Plane



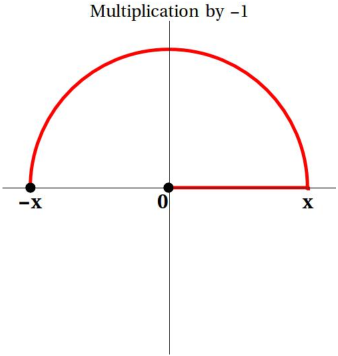
The positive number x is marked to the right of the origin on the number line.

$$1 \times x = x$$

We can regard 1 as an operator acting on x

It is noteworthy that $1 \times x = x \times 1$.

From Number Line to Complex Plane



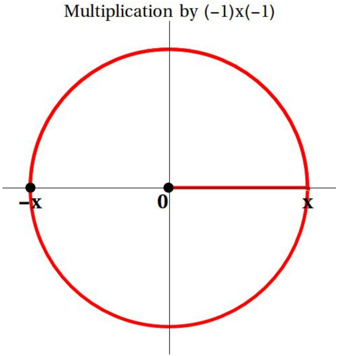
The number -1 operates on x by rotating it through 180°.

$$(-1) \times x = -x$$

Multiplying by -1 means a rotation through π radians.

Positive numbers become negative and vice versa.

From Number Line to Complex Plane



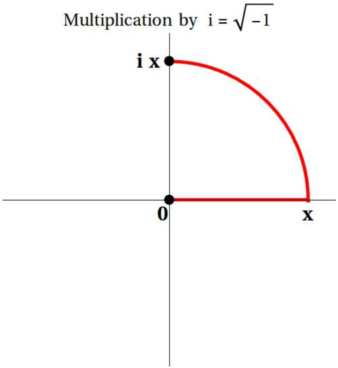
Multiplying twice by -1 gives a rotation through 360°.

$$(-1) \times (-1) \times x = x$$

Multiplying twice by -1 means rotation through 2π.

Both positive and negative numbers unchanged.

From Number Line to Complex Plane



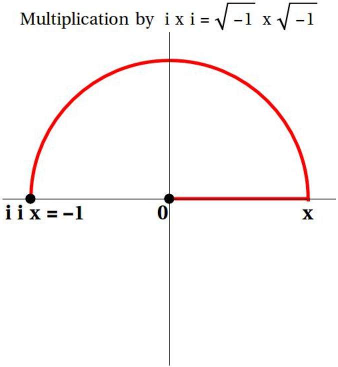
Now imagine rotating through 90° **or** $\pi/2$ **radians.**

This brings us away from the real line.

We write the operator that rotates through $\pi/2$ **as** i**.**

What if we do this twice? We go through π **radians!**

From Number Line to Complex Plane



Multiply twice by i

This means a rotation of 90° followed by another rotation of 90°.

So operating twice with i equals once with -1.

Therefore $i \times i = -1$ which means $i = \sqrt{-1}$.

Rotations about the Origin

$$\begin{aligned} i &= \sqrt{-1} \\ i^2 &= -1 \\ i^3 &= -i \\ i^4 &= 1 \end{aligned}$$

$i^0 = 1 \implies$ **Rotate 0°**

$i^1 = i \implies$ **Rotate 90°**

$i^2 = -1 \implies$ **Rotate 180°**

$i^3 = -i \implies$ **Rotate 270°**

$i^4 = 1 \implies$ **Rotate 360°**

Imaginary Numbers

We can get to any place on the vertical axis by multiplying a real number y by i, written iy.

Numbers on the vertical axis are called imaginary numbers.

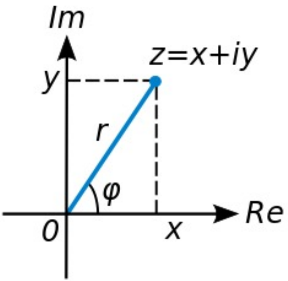
This is unfortunate. They are every bit as real as "real numbers".

Complex Numbers

We can combine a real number x and an imaginary number iy to give a complex number

$$z = x + iy$$

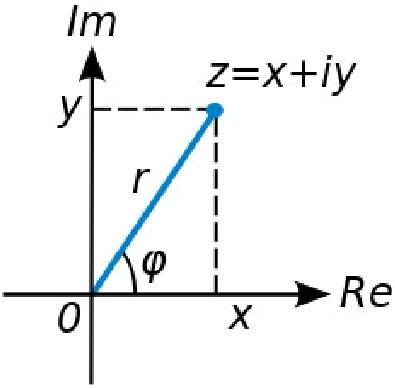
The complex number $z = x + iy$ is represented in the complex plane by the point (x, y).



Every point in the plane gives a complex number.
Every complex number gives a point in the plane.

The Complex Plane

This is the complex plane or Gaussian plane.



It is occasionally called *the Argand diagram.*

Calculating with Complex Numbers

Addition of complex numbers is very simple: let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

Multiplication is also simple: we just apply the rule for multiplying i by itself:

$$\begin{aligned} z_1 \, z_2 &= (x_1 + iy_1) \times (x_2 + iy_2) \\ &= x_1 x_2 + x_1 iy_2 + iy_1 x_2 + iy_1 iy_2 \\ &= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2) \end{aligned}$$

Now we have extended the number system:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C} \,.$$

Outline

Euler’s Fabulous Formula

Several surveys have been carried out to determine the most beautiful mathematical formula.

The consistent winner has been the formula

$$e^{i\pi} + 1 = 0$$

First derived by Leonhard Euler.

This is a remarkable result. It combines in one simple formula the five most important numbers

$$0 \qquad 1 \qquad \pi \qquad e \qquad i$$

We will now show where the result comes from.

Taylor Series for Sin, Cos and Exp

Recall the Taylor series for $\sin x$, $\cos x$ **and** $\exp x$**:**

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\ \exp x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \end{aligned}$$

These series are valid for complex arguments.
For example,

$$\exp z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$$

We now substitute the value $z = i\theta$ into the series

$$\exp z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$$

$$\begin{aligned}
\exp(i\theta) &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots \\
&= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \cdots \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) \\
&\quad + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right) \\
&= \cos(\theta) + i\sin(\theta)
\end{aligned}$$

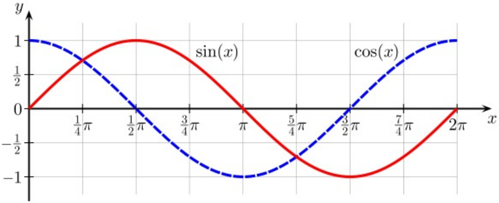
Again,

$$\exp(i\theta) = \cos(\theta) + i\sin(\theta)$$

For $\theta = \pi$ this is

$$\exp(i\pi) = \cos(\pi) + i\sin(\pi)$$

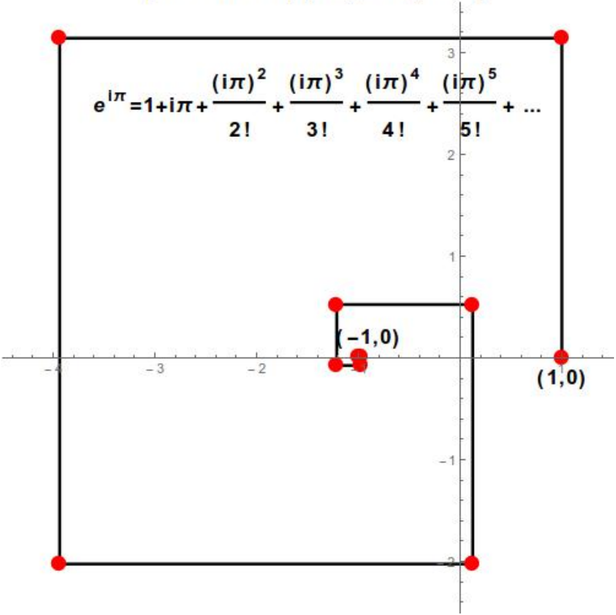
But $\cos(\pi) = -1$ and $\sin(\pi) = 0$:



Therefore, we have $\exp(i\pi) = -1$ or

$$\exp(i\pi) + 1 = 0$$

Spiral of exp(iπ) to (−1,0)



First point : 1

Second point : $1 + i\pi$

Third point : $\left(1 - \frac{\pi^2}{2!}\right) + i\pi$

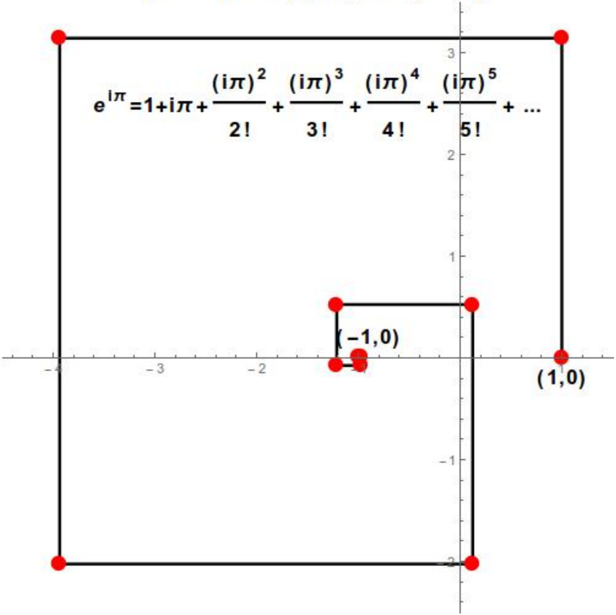
Fourth point : $\left(1 - \frac{\pi^2}{2!}\right) + i\left(\pi - \frac{\pi^3}{3!}\right)$

Fifth point : $\left(1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!}\right) + i\left(\pi - \frac{\pi^3}{3!}\right)$

Sixth point : $\left(1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!}\right) + i\left(\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!}\right)$

Seventh point : $\left(1 - \frac{pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!}\right) + i\left(\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!}\right)$

Spiral of exp(iπ) to (–1,0)



Outline

Defining an Iterative Function

We define a sequence of complex numbers

$$z_0 = 0\,, \qquad z_{n+1} = z_n^2 + c$$

where c is a (constant) complex parameter.

This gives the sequence

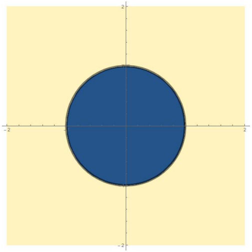
$$\{0,\ c,\ c^2 + c,\ c^4 + 2c^3 + c^2 + c, \ldots\}$$

Does this sequence converge or diverge?

It depends on the value of the parameter c.

Simple Example of Escape Region

We define a geometric series by an iterative process:

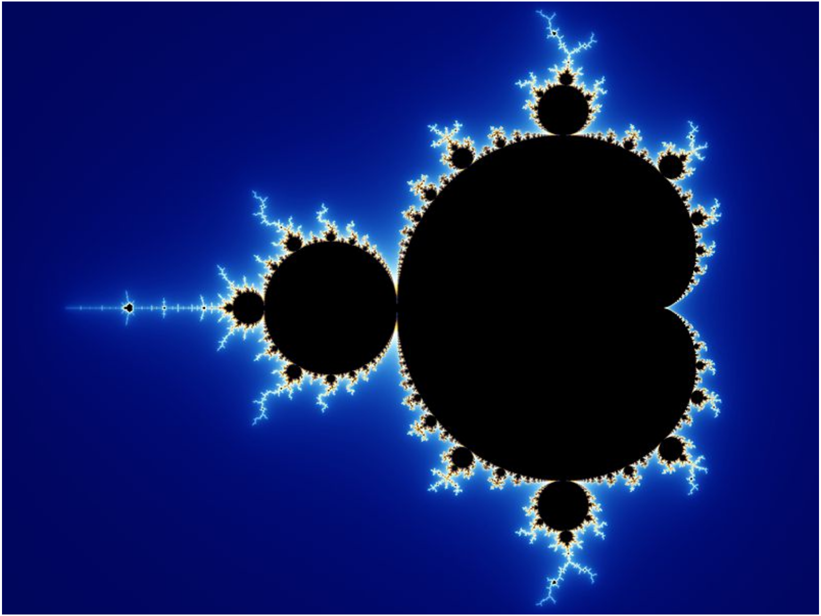


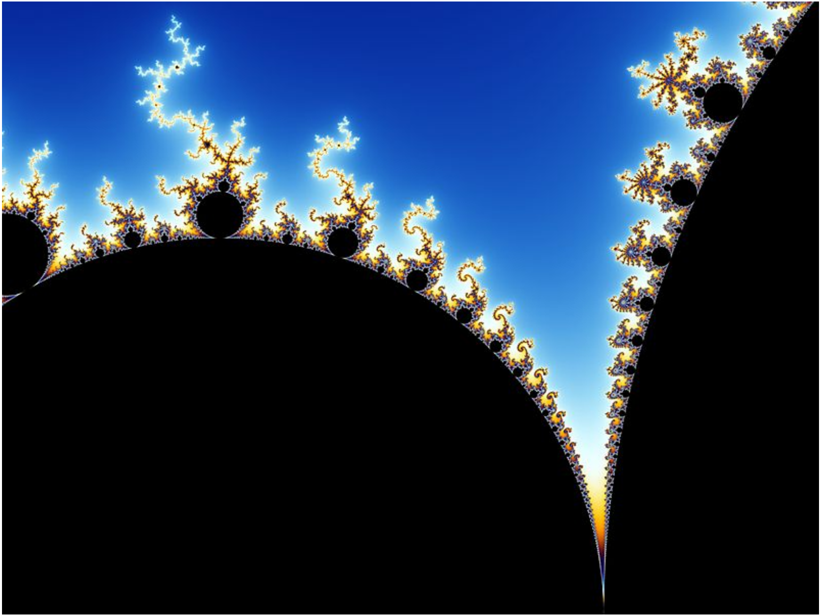
$$z_0 = c$$
$$z_{n+1} = c\, z_n$$

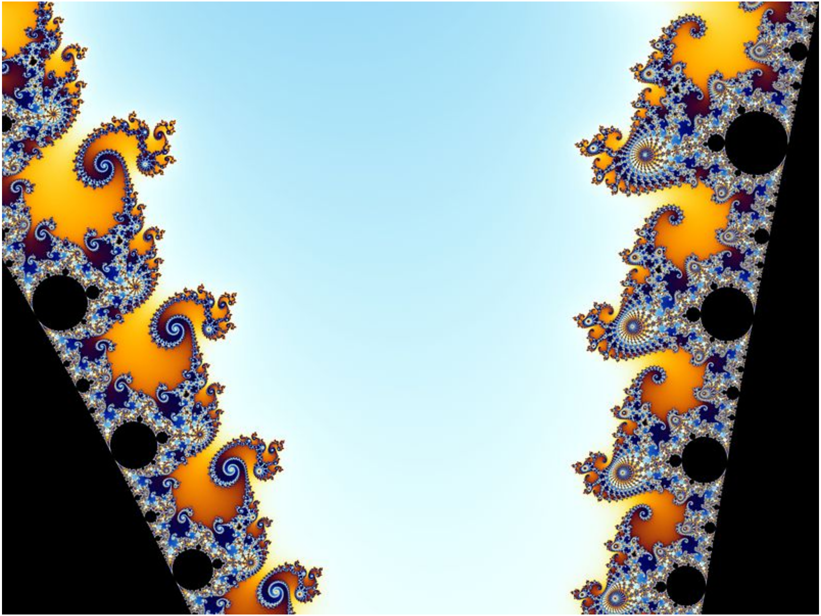
For $|c| < 1$ the sequence $\{z_n\}$ converges to zero.

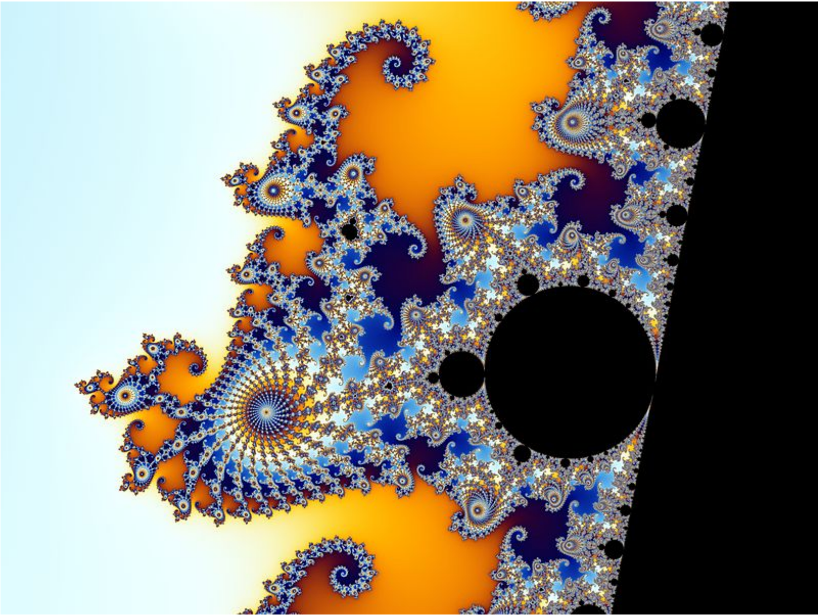
For $|c| > 1$ the sequence $\{z_n\}$ diverges [*escapes*] to infinity.

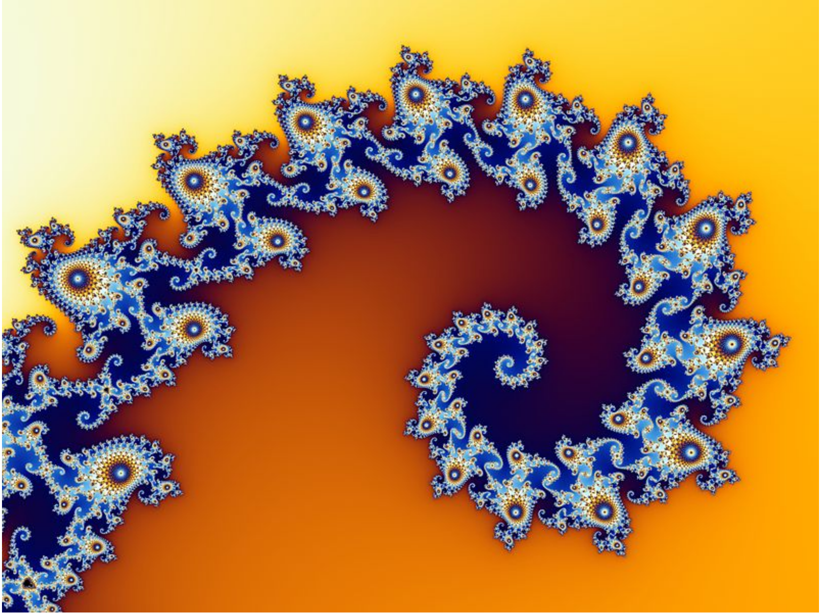
We can colour-code the escape region to indicate how fast the sequence diverges.

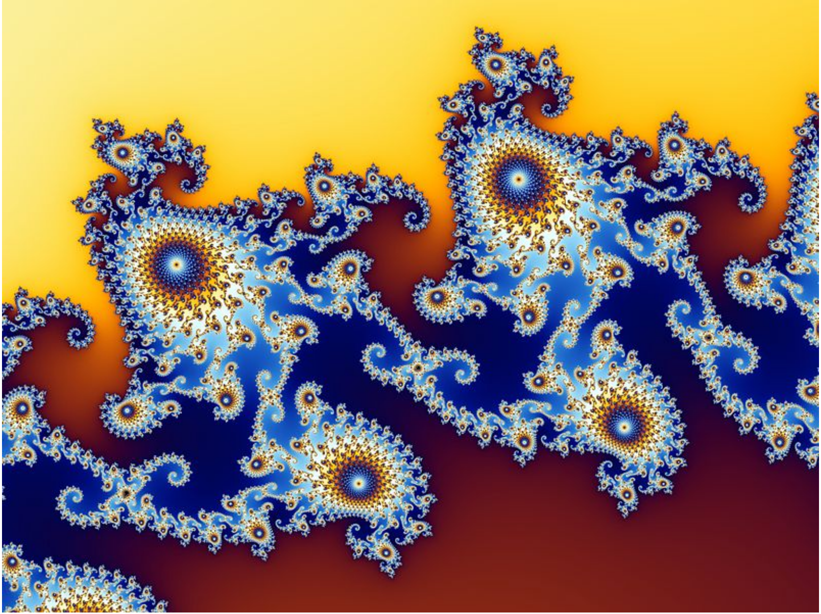


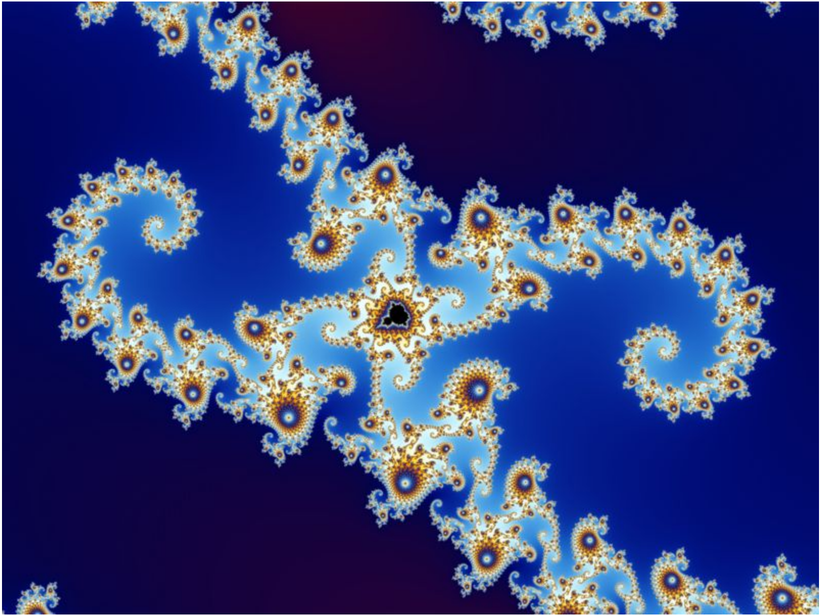


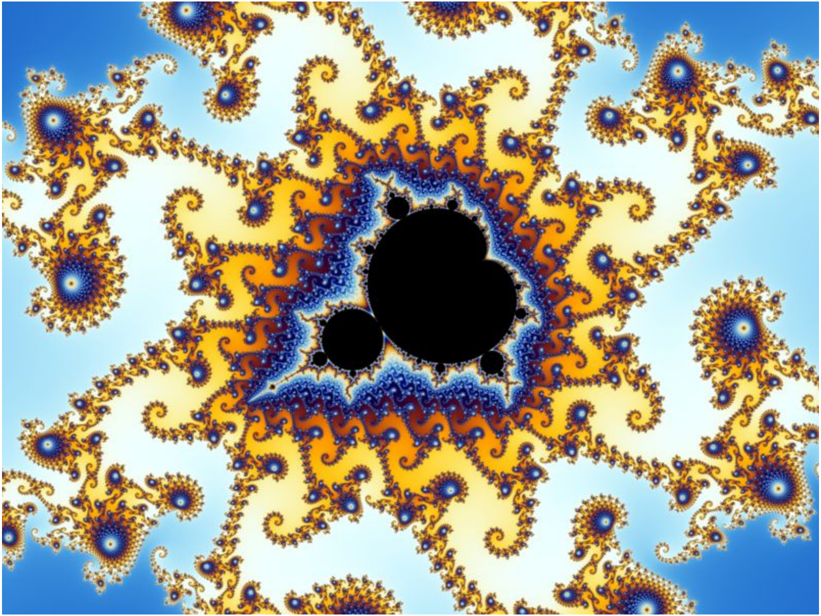


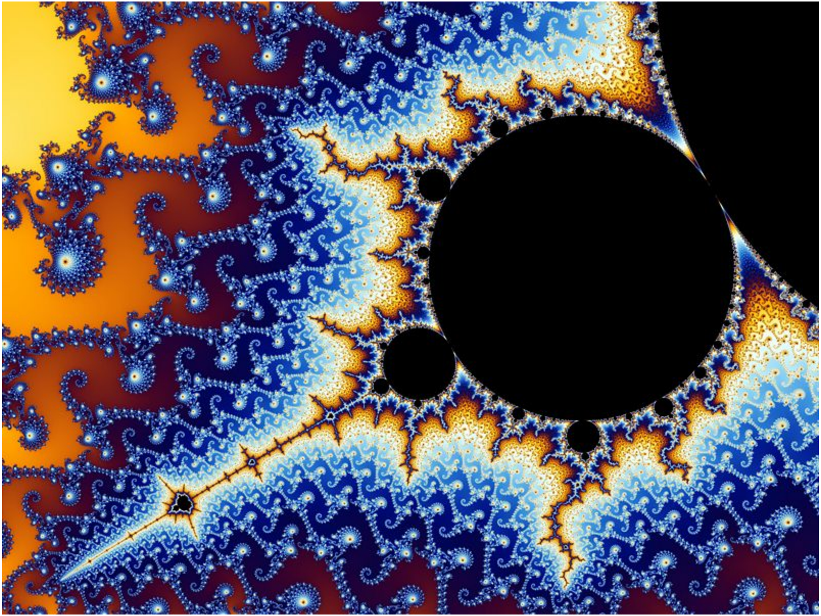


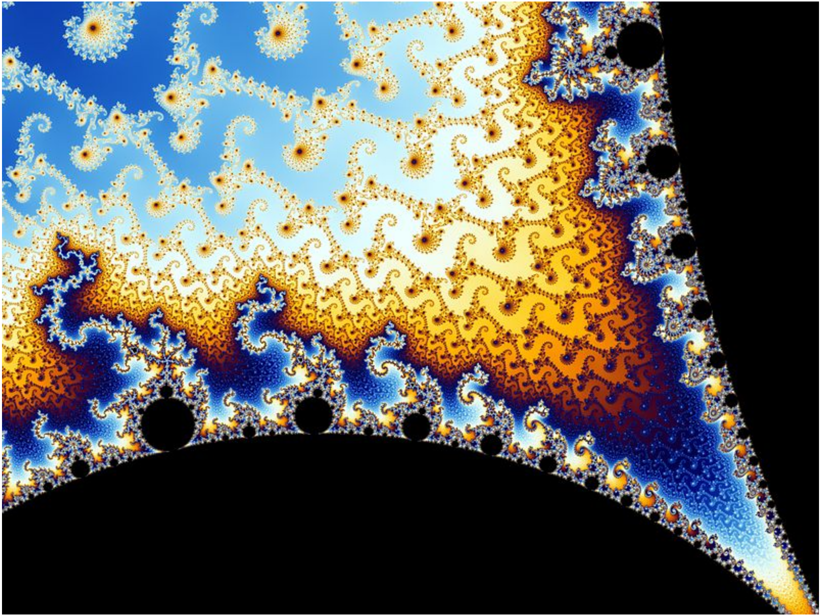


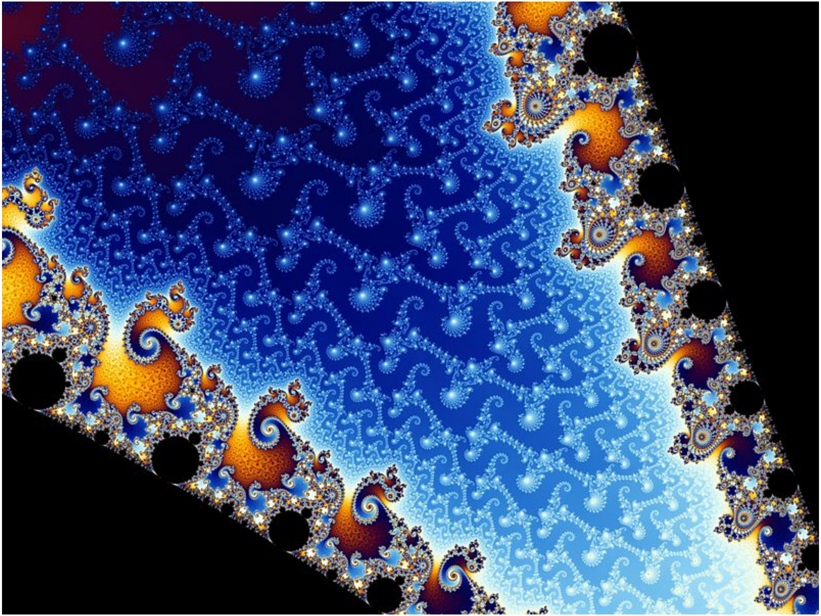


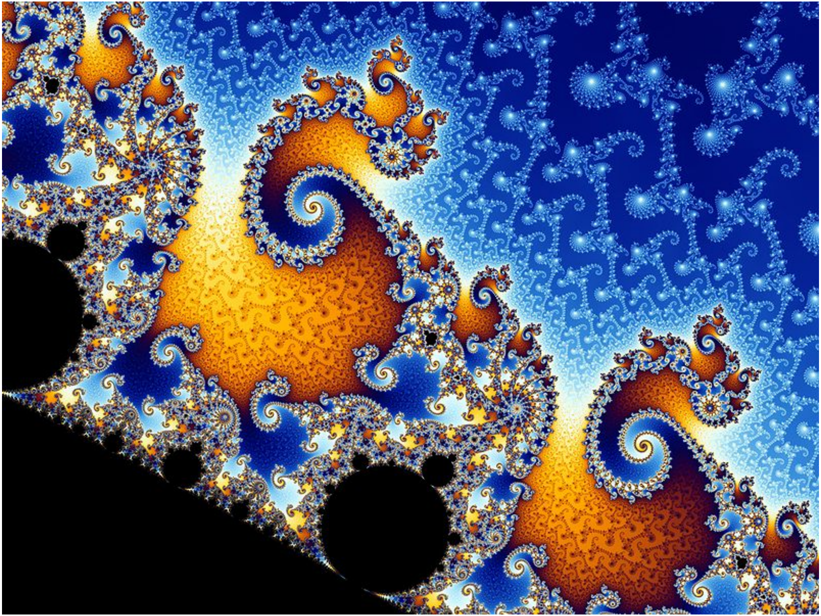


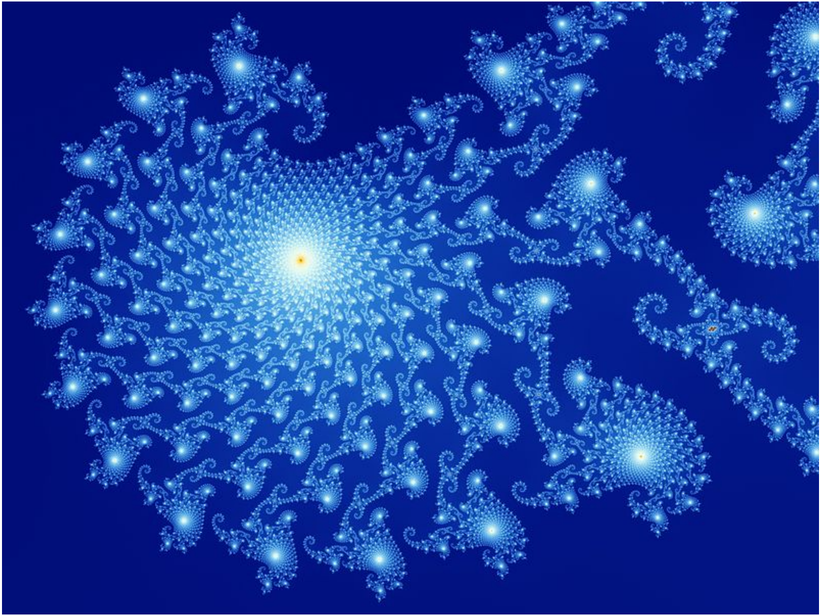


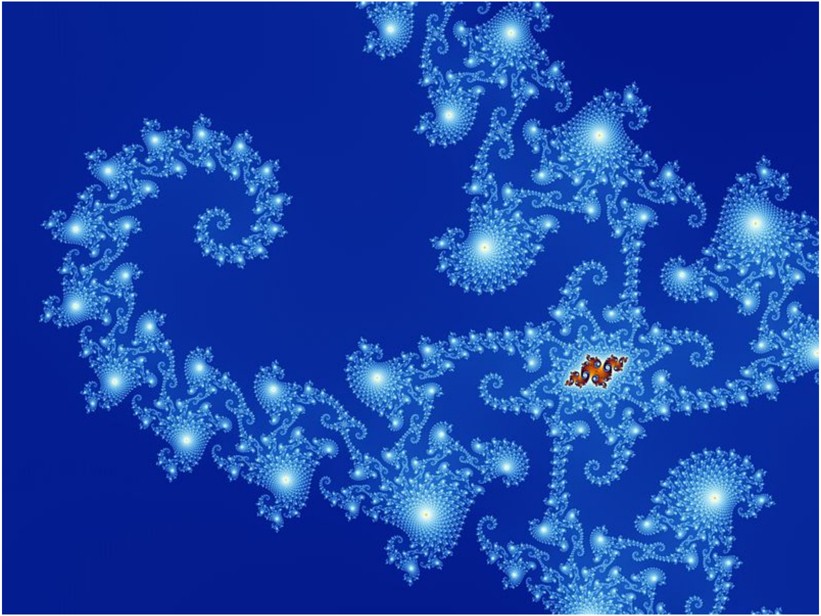


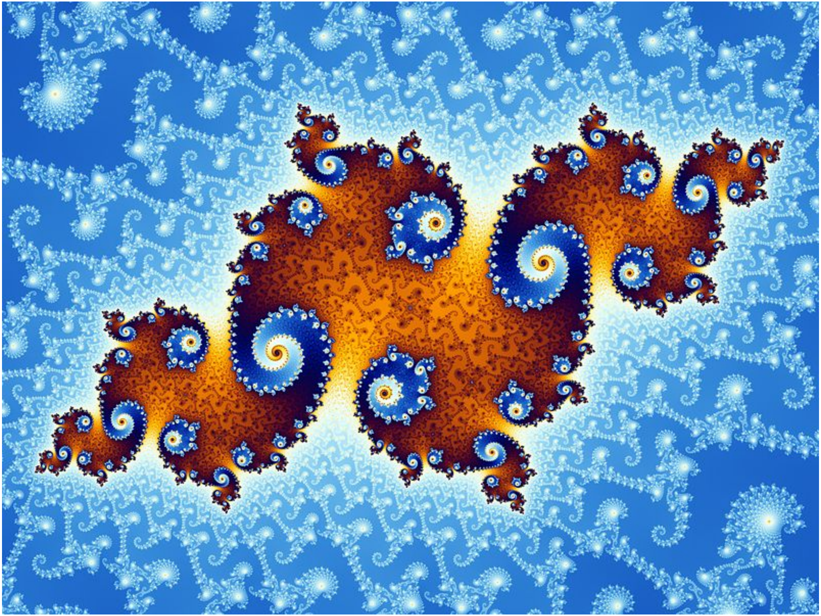












“Is Minic Freagra ar Imeall”

All this AweSum Majesty comes from the simple iterative process in the complex plane:

$$z_{n+1} = z_n^2 + c$$

We plot the “escape time” as a function of the complex valued parameter c.

In the black region, $\{z_n\}$ remains bounded. Outside this region, it diverges to infinity.

The rate of divergence depends on c. The plots are colour-coded according to this rate.

Thank you