AweSums:

The Majesty of Mathematics

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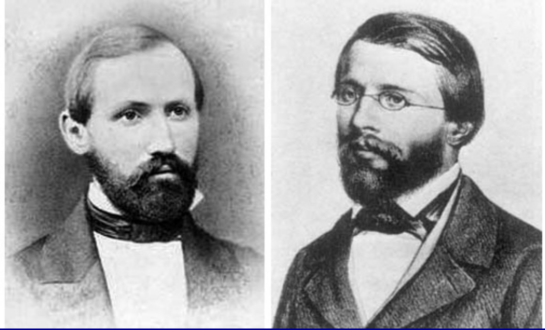
Evening Course, UCD, Autumn 2016

Outline

Outline

AweSums: The Majesty of Maths



Bernhard Riemann (1826-66)

AweSums: The Majesty of Maths

We aim to get a flavour of the Riemann Hypothesis.

It involves the zeros of the "Zeta function":

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

So, we need to talk about several new topics.

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So, we need to talk about several new topics.

In this lecture, we will look at trigonometric functions.

Outline

Infinite Series

A **series** is an infinite sum of numbers indexed by the natural numbers:

$$S = a_1 + a_2 + a_3 + \cdots + a_n + \ldots$$

We write this using sigma-notation:

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The convergence of S depends on the terms a_n.

There is a wide range of convergence tests.

A Geometric Series

We looked at the geometric series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

where each term is half the previous one.

The sum gets closer and closer to 2
as n becomes larger and larger.

The series converges to 2.

A Geometric Series

More generally, we write the geometric series

$$S = 1 + x + x^2 + x^3 + x^4 + \cdots$$

Clearly, if $|x| < 1$ the terms are getting smaller whereas if $|x| > 1$ the terms are getting larger.

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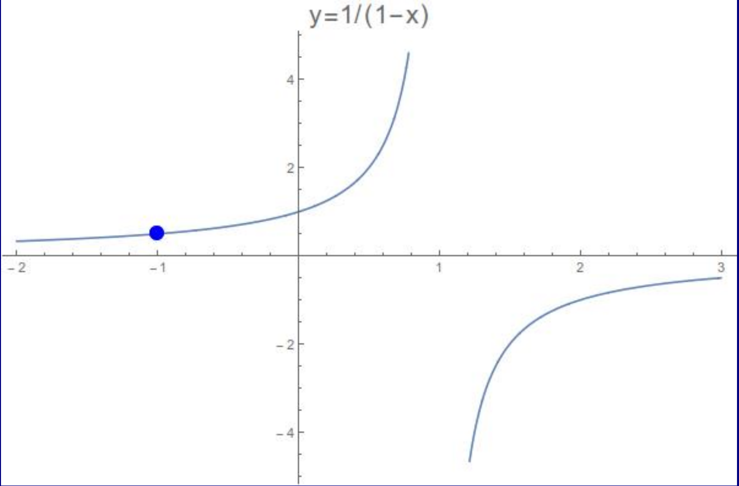
To demonstrate this, subtract $x\,S$ from S:

$$\begin{array}{rcl} S & = & 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots \\ -x\,S & = & \quad - x - x^2 - x^3 - x^4 - x^5 - \cdots \\ \hline (1-x)\,S & = & 1 \end{array}$$

So $S = \frac{1}{1-x}$

$f(x) = \frac{1}{1-x}$



This function "blows up" at $x = 1$ but $y = \frac{1}{2}$ at $x = -1$.

Analytical Continuation

We define a function $f(x)$ by the geometric series

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$$f(x) = \frac{1}{1-x}\,, \qquad |x| < 1$$

This function also has a meaning for $|x| > 1$.
We have effectively extended the function
beyond the range $-1 < x < +1$.

This process is called analytic continuation.
It is used to extend Riemann's zeta-function.

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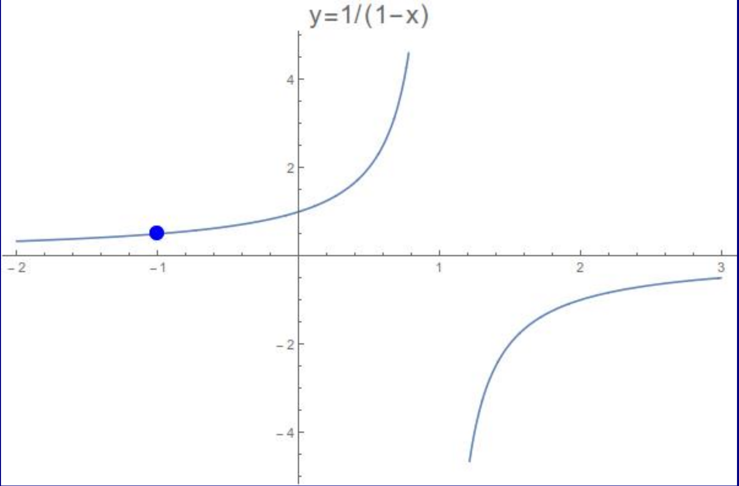
If $x = -1$ we get the alternating sum

$$1 - 1 + 1 - 1 + 1 \cdots$$

The partial sums alternate between 1 and 0.
The series does not converge.

$f(x) = \frac{1}{1-x}$



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This can be made rigorous: the Cesàro sum is the limit of the mean of the partial sums of the series.

The Harmonic Series

Let's look at a few other interesting infinite series.

We defined the **harmonic series** as

$$H = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

The sum becomes larger without limit: it diverges!

$$H_n \sim \log n + \gamma$$

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We defined the alternating harmonic series:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \log 2$$

which is *conditionally convergent.*

The Inverse Prime Series

The sum of the inverses of the prime numbers

$$P = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots = \sum_{n=1}^{\infty} \frac{1}{p_n}$$

diverges, but very, very slowly.

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It can be shown that

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where M is known as the Mertens number.

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Even for $p \approx 10^{100}$, the sum is less than 6.

The Leibniz Series for π

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$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

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The series is of no practical use in evaluating π.

Outline

Distraction 8: The Galway Girl

http://www.skibbereeneagle.ie/...
.../ireland/galway-girl/

The Galway Girl

I want to see my girlfriend in Galway.
But I'm shy! Will I ever get there?

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- I travel half-way to Galway.
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- But, half-way back, I regain courage.
- I travel half the distance to Galway.
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- Back and forth, hither and thither . . .

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Is there any hope, or will my love remain unrequited?

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Let my distance from Dublin be $x_0 = 0$ at the outset.
Let the distance from Dublin to Galway be 1.

After an even number of stages, the distance is x_{2n}.

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The next two stages are:

$$x_{2n+1} = \tfrac{1}{2}(x_{2n} + 1) \qquad \textbf{and} \qquad x_{2n+2} = \tfrac{1}{2}x_{2n+1}$$

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Suppose that this sequence converges to X. Then

$$X = \tfrac{1}{4}X + \tfrac{1}{4} \qquad \textbf{which means} \qquad X = \tfrac{1}{3}$$

Does this mean that the sequence $\{x_n\}$ converges?

The Galway Girl

We found that $x_{2n+1} = \frac{1}{2}(x_{2n} + 1)$ **and** $x_{2n+2} = \frac{1}{4}x_{2n} + \frac{1}{4}$**.**

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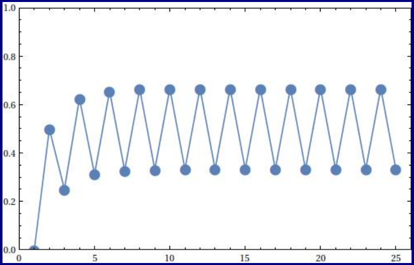

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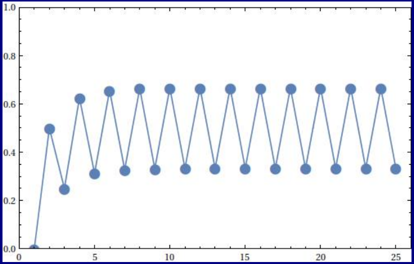

I am doomed to spend my life travelling back and forth between Kinnegad and Ballinasloe.

The Galway Girl: Plan B

I get more courageous. I have a new plan.

Each time I travel half-way to Galway,
I return half the distance I have just travelled.

Will I ever get to see the Galway Girl?

The Galway Girl: Plan B

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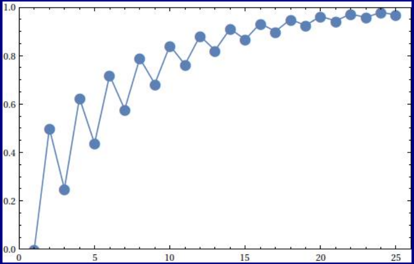
The sequence $\{x_n\}$ converges to 1?

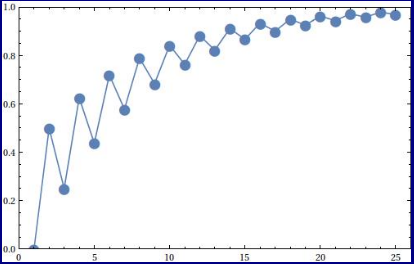
The Galway Girl: Plan B

So, do I get to see the girl?



The Galway Girl: Plan B

So, do I get to see the girl?



How far do I travel? Assuming constant speed, How long does it take to reach Galway?

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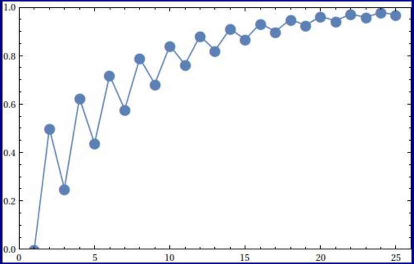


How far do I travel? Assuming constant speed, How long does it take to reach Galway?

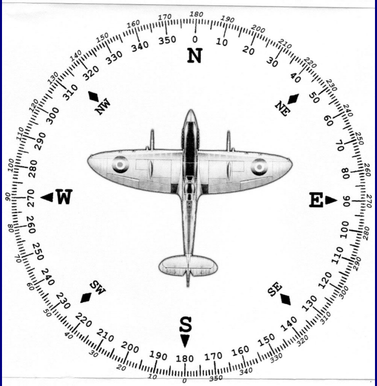
Remember Zeno.

Total distance 3 units.

Outline

Angular Measure: Degrees

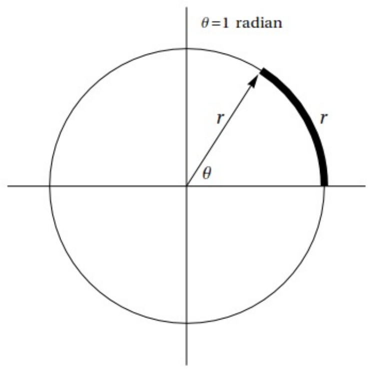
The Babylonians divided the circle into 360 degrees.



- Their number system used the base 60.
- It is easy to draw a hexagon in a circle.
- There are **about** 360 days in a year.

We still use the 360° division of the circle today.

Angular Measure: Radians



A **radian** is an angle in a circle with arc length equal to its radius.

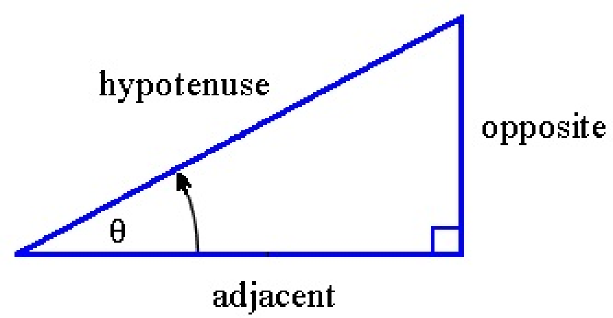
$$1\,\mathrm{r}ad \approx 57.3^\circ$$

There are 2π radians in a full circle.

A right angle is both 90° and $\pi/2$ radians.

Radian measure is the standard method of measuring angles in mathematics.

Sides of a Right Triangle



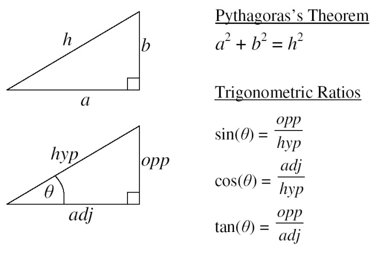
The side opposite the right angle is the hypotenuse.

We choose another angle θ.
The side close to θ is the adjacent side.

The side farthest from θ is the opposite side.

Sine, Cosine and Tangent

The trigonometric functions are ratios of sides of a right-angled triangle.



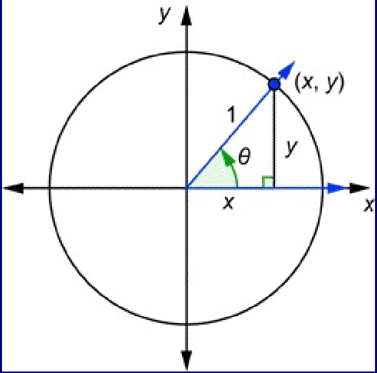
$$\sin(\theta) = \frac{\textit{opposite}}{\textit{hypotenuse}}$$

$$\cos(\theta) = \frac{\textit{adjacent}}{\textit{hypotenuse}}$$

$$\tan(\theta) = \frac{\textit{opposite}}{\textit{adjacent}}$$

The usual mnemonic is SohCahToa or SOH CAH TOA

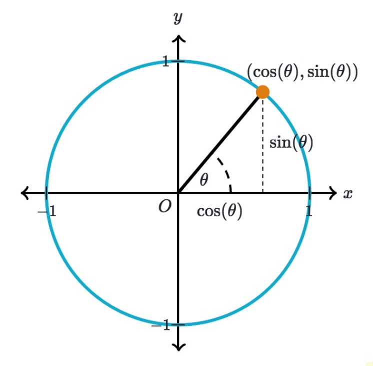
Unit Circle



By Pythagoras' Theorem,

$$x^2 + y^2 = 1$$

Unit Circle



On the unit circle we have

$$x = \cos(\theta)$$
$$y = \sin(\theta)$$

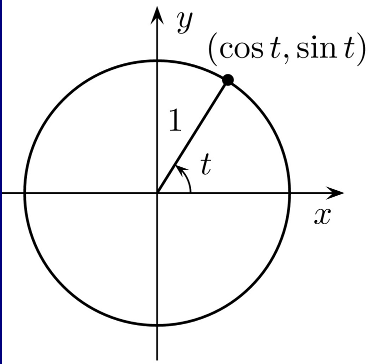
By Pythagoras' Theorem,

$$x^2 + y^2 = 1$$

Therefore

$$(\cos\theta)^2 + (\sin\theta)^2 = 1$$

Unit Circle



Now let us denote the angle by t.

This is to suggest time.

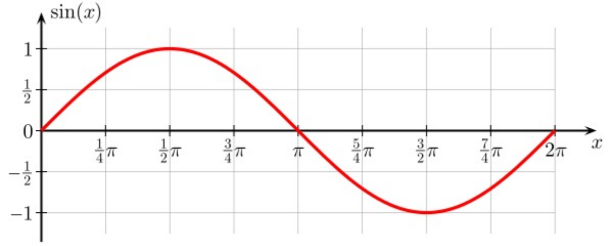
How do x and y vary as the time passes?

Animation of a Sine Wave

Animation showing how the sine function (in red) $y = \sin(\theta)$ is graphed from the y-coordinate (red dot) of a point on the unit circle (in green) at an angle of θ in radians.

https://en.wikipedia.org/wiki/Sine/

Sine Waves over One Period



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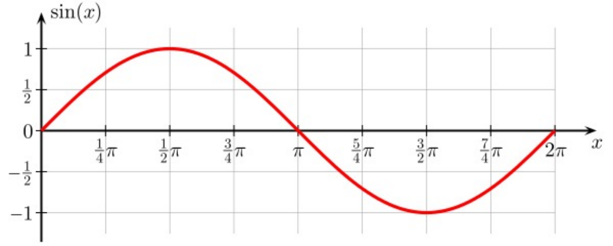
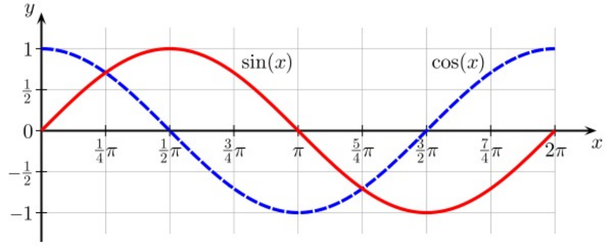


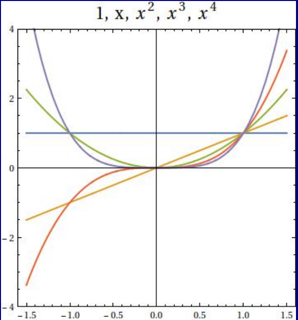


Outline

Introduce Polynomials on BB

Outline the properties and graphs of simple polynomials on the blackboard.

Basis Functions for Approximation



Many functions can be approximated by a series of polynomial functions.

Here we plot the functions

$$1 \quad x \quad x^2 \quad x^3 \quad x^4$$

used as basis functions.

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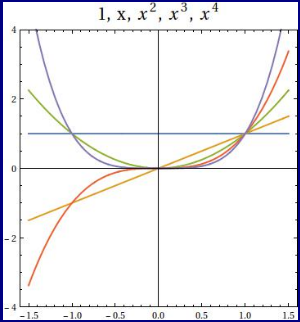

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Most functions $f(x)$ can be approximated by a simple polynomial function of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

Polynomial Approximation. Taylor Series

Any "reasonable function" $f(x)$ can usually be approximated by a simple polynomial function

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$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

Sometimes we can find the roots of the polynomial; that is, the values of x for which it is zero.

Then we are able to write the polynomial as

$$p(x) = a_n(x - x_1)(x - x_2)(x - x_3)\cdots(x - x_n)$$

It is simple to sketch the graph of this function.

Taylor Series for Sine Wave

The Taylor series for $\sin x$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

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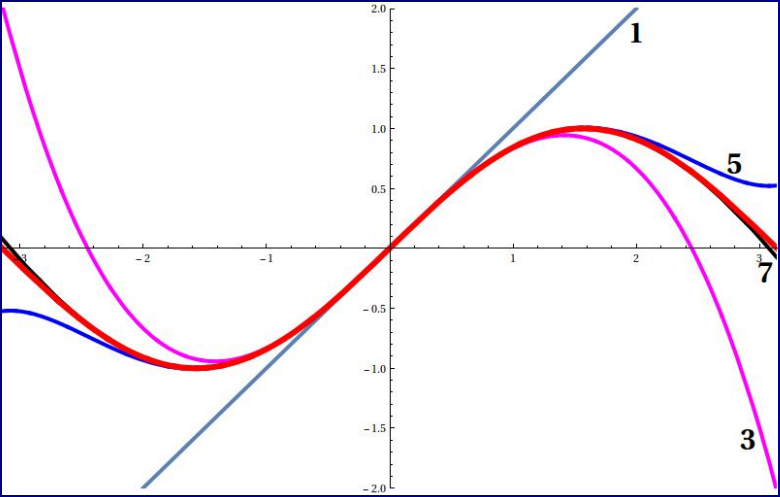
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

We truncate to get a sequence of polynomials:

$$\begin{aligned} p_1(x) &= x \\ p_3(x) &= x - \frac{x^3}{3!} \\ p_5(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} \\ p_7(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \end{aligned}$$

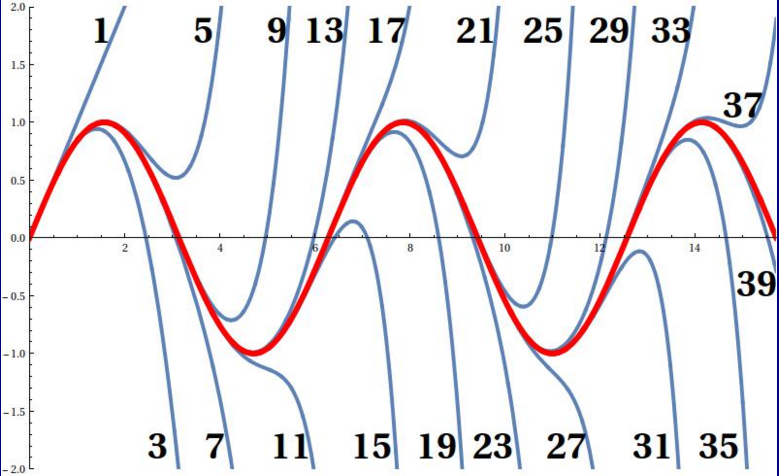
They approximate $\sin x$ **better with increasing order.**

Polynomial Approximation to Sine Wave



$p_7(x)$ **is a good fit over a full wavelength.**

Polynomial Approximation to Sine Wave



$p_{39}(x)$ **fits well over five wavelengths.**

Thank you