AweSums:

The Majesty of Mathematics

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Outline

Outline

AweSums: The Majesty of Maths

Bernhard Riemann (1826-66)

AweSums: The Majesty of Maths

We aim to get a flavour of the Riemann Hypothesis.

It involves the zeros of the "Zeta function":

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

So, we need to talk about several *new topics.*

In this lecture, we will look at trigonometric functions.

Outline

Infinite Series

A *series* is an infinite sum of numbers indexed by the natural numbers:

$$S = a_1 + a_2 + a_3 + \cdots + a_n + \dots$$

We write this using sigma-notation:

$$S = \sum_{n=1}^{\infty} a_n$$

The convergence of S depends on the terms a_n.

There is a wide range of convergence tests.

A Geometric Series

We looked at the geometric series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

where each term is half the previous one.

The sum gets closer and closer to 2
as n becomes larger and larger.

The series converges to 2.

A Geometric Series

More generally, we write the geometric series

$$S = 1 + x + x^2 + x^3 + x^4 + \cdots$$

Clearly, if $|x| < 1$ the terms are getting smaller whereas if $|x| > 1$ the terms are getting larger.

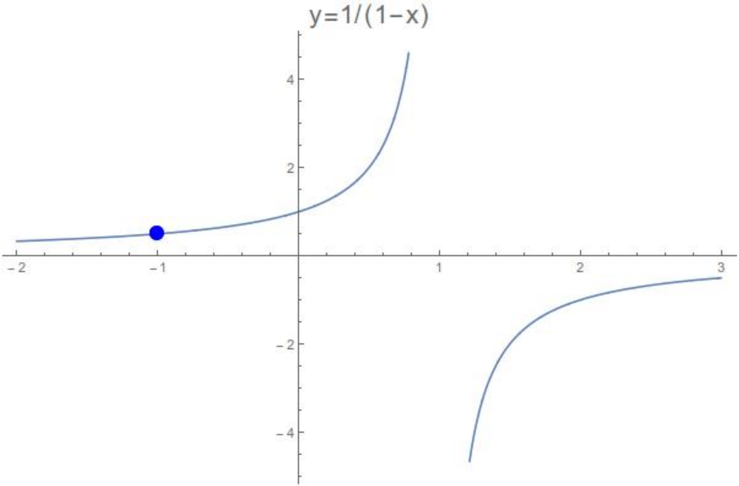
We can show that for $|x| < 1$ the sum S is $\dfrac{1}{(1-x)}$

To demonstrate this, subtract $x\,S$ from S:

$$\begin{array}{rcl} S & = & 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots \\ -x\,S & = & \quad - x - x^2 - x^3 - x^4 - x^5 - \cdots \\ \hline (1-x)\,S & = & 1 \end{array}$$

So $S = \frac{1}{1-x}$

$f(x) = \frac{1}{1-x}$



This function "blows up" at $x = 1$ but $y = \frac{1}{2}$ at $x = -1$.

Analytical Continuation

We define a function $f(x)$ by the geometric series

$$f(x) = 1 + x + x^2 + x^3 + x^4 + \cdots$$

which converges for $|x| < 1$ and diverges for $|x| \geq 1$.

We showed that for $|x| < 1$ the sum is $\frac{1}{1-x}$. Therefore

$$f(x) = \frac{1}{1-x}\,, \qquad |x| < 1$$

This function also has a meaning for $|x| > 1$.
We have effectively extended the function beyond the range $-1 < x < +1$.

This process is called *analytic continuation.*
It is used to extend Riemann's zeta-function.

Special Cases of the Geometric Series

$$S = 1 + x + x^2 + x^3 + x^4 + \cdots$$

If $x = \frac{1}{2}$ we get the familiar series converging to 2:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$$

If $x = 1$ we get a divergent sum of ones

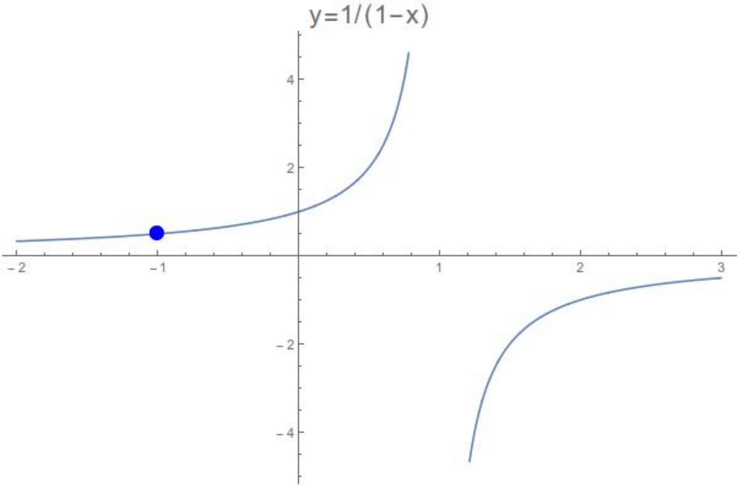
$$1 + 1 + 1 + 1 + 1 + 1 + \cdots$$

If $x = -1$ we get the alternating sum

$$1 - 1 + 1 - 1 + 1 \cdots$$

The partial sums alternate between 1 and 0.
The series does not converge.

$f(x) = \frac{1}{1-x}$



This function "blows up" at $x = 1$ but $y = \frac{1}{2}$ at $x = -1$.

Curious Interpretation. Summation

If $x = -1$ we get the alternating sum

$$1 - 1 + 1 - 1 + 1 \cdots$$

The partial sums alternate between 1 and -1.
The series does not converge.

But let us consider the sequence of partial sums:

$$s_1 = 1 \quad s_2 = 0 \quad s_3 = 1 \quad s_4 = 0 \quad s_5 = 1 \cdots$$

In a curious way, this suggests an *average value* of $\frac{1}{2}$.

This can be made rigorous: the Cesàro sum is the *limit of the mean* of the partial sums of the series.

The Harmonic Series

Let's look at a few other interesting infinite series.

We defined the harmonic series as

$$H = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \cdots$$

The sum becomes larger without limit: it diverges!

$$H_n \sim \log n + \gamma$$

We defined the alternating harmonic series:

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \cdots = \log 2$$

which is *conditionally convergent.*

The Inverse Prime Series

The sum of the inverses of the prime numbers

$$P = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots = \sum_{n=1}^{\infty} \frac{1}{p_n}$$

diverges, but very, very slowly.

It can be shown **that**

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{p} \sim \log \log p + M$$

where M is known as the Mertens number.

Even for $p \approx 10^{100}$, the sum is less than 6.

The Leibniz Series for π

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots = \frac{\pi}{4}$$

This is also called the Madhava-Leibniz series.

The following series was discovered by the (14th cen.) Indian mathematician Madhava of Sangamagrama

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

The Leibniz formula follows by setting $x = 1$**.**

The series is of no practical use in evaluating π**.**

Outline

Distraction 8: The Galway Girl

`http://www.skibbereeneagle.ie/...`
`.../ireland/galway-girl/`

The Galway Girl

I want to see my girlfriend in Galway.
But I'm shy! Will I ever get there?

- I travel half-way to Galway.
- Losing my nerve, I return towards Dublin.
- But, half-way back, I regain courage.
- I travel half the distance to Galway.
- Then I travel half the distance to Dublin.
- Back and forth, hither and thither . . .

Is there any hope, or will my love remain unrequited?

The Galway Girl

Let my distance from Dublin be $x_0 = 0$ at the outset.
Let the distance from Dublin to Galway be 1.

After an even number of stages, the distance is x_{2n}.

The next two stages are:

$$x_{2n+1} = \tfrac{1}{2}(x_{2n} + 1) \qquad \text{and} \qquad x_{2n+2} = \tfrac{1}{2} x_{2n+1}$$

Therefore, $x_{2n+2} = \frac{1}{4} x_{2n} + \frac{1}{4}$

Suppose that this sequence converges to X. Then

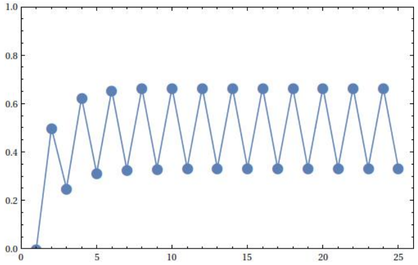
$$X = \tfrac{1}{4} X + \tfrac{1}{4} \qquad \text{which means} \qquad X = \tfrac{1}{3}$$

Does this mean that the sequence $\{x_n\}$ converges?

The Galway Girl

We found that $x_{2n+1} = \frac{1}{2}(x_{2n} + 1)$ **and** $x_{2n+2} = \frac{1}{4}x_{2n} + \frac{1}{4}$**.**

But this sequence does not converge. It oscillates, in a limit cycle, between $x = \frac{1}{3}$ **and** $x = \frac{2}{3}$**.**



I am doomed to spend my life travelling back and forth between Kinnegad and Ballinasloe.

The Galway Girl: Plan B

I get more courageous. I have a new plan.

Each time I travel half-way to Galway,
I return half the distance I have just travelled.

Will I ever get to see the Galway Girl?

The Galway Girl: Plan B

Let my distance from Dublin be $x_0 = 0$ at the outset.
Let the distance from Dublin to Galway be 1.

The next two stages are:

$$x_{2n+1} = \tfrac{1}{2}(x_{2n} + 1) \qquad \text{and} \qquad x_{2n+2} = \tfrac{1}{2}(x_{2n} + x_{2n+1})$$

Therefore, $x_{2n+2} = \tfrac{3}{4}x_{2n} + \tfrac{1}{4}$

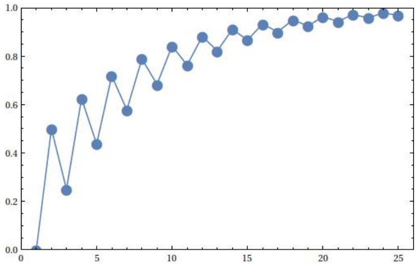
Suppose that this sequence converges to X. Then

$$X = \tfrac{3}{4}X + \tfrac{1}{4} \qquad \text{which means} \qquad X = 1$$

The sequence $\{x_n\}$ converges to 1?

The Galway Girl: Plan B

So, do I get to see the girl?



How far do I travel? Assuming constant speed,
How long does it take to reach Galway?

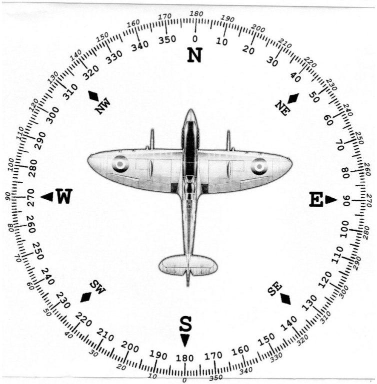
Remember Zeno.

Total distance 3 units.

Outline

Angular Measure: Degrees

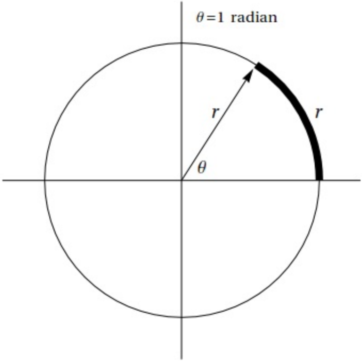
The Babylonians divided the circle into 360 degrees.



- **Their number system used the base 60.**
- **It is easy to draw a hexagon in a circle.**
- **There are *about* 360 days in a year.**

We still use the 360° **division of the circle today.**

Angular Measure: Radians



A radian is an angle in a circle with arc length equal to its radius.

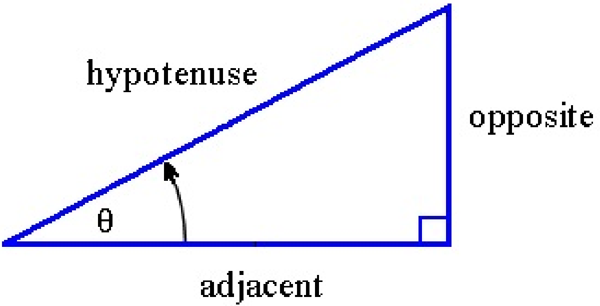
$$1\,\mathrm{r}ad \approx 57.3^\circ$$

There are 2π radians in a full circle.

A right angle is both 90° and $\pi/2$ radians.

Radian measure is the standard method of measuring angles in mathematics.

Sides of a Right Triangle



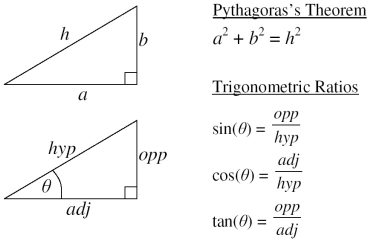
The side opposite the right angle is *the hypotenuse.*

We choose another angle θ.
The side close to θ is the *adjacent side.*

The side farthest from θ is the *opposite side.*

Sine, Cosine and Tangent

The trigonometric functions are ratios of sides of a right-angled triangle.



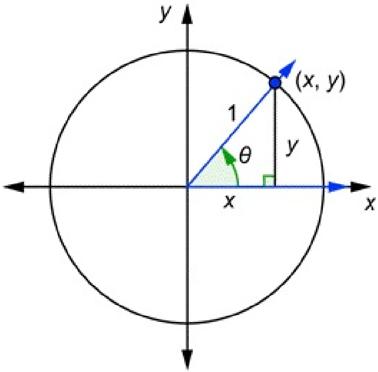
$$\sin(\theta) = \frac{\textit{opposite}}{\textit{hypotenuse}}$$

$$\cos(\theta) = \frac{\textit{adjacent}}{\textit{hypotenuse}}$$

$$\tan(\theta) = \frac{\textit{opposite}}{\textit{adjacent}}$$

The usual mnemonic is SohCahToa or SOH CAH TOA

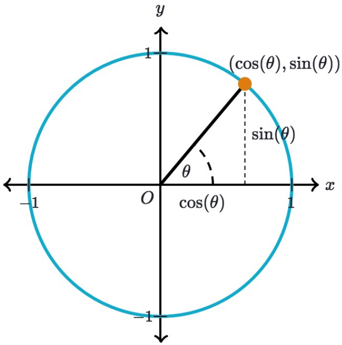
Unit Circle



By Pythagoras' Theorem,

$$x^2 + y^2 = 1$$

Unit Circle



On the unit circle we have

$$x = \cos(\theta)$$
$$y = \sin(\theta)$$

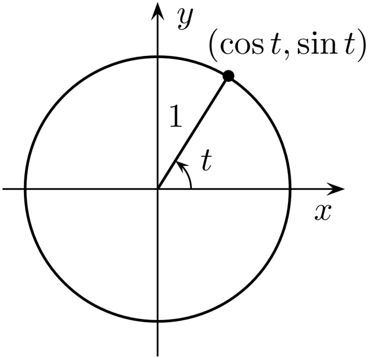
By Pythagoras' Theorem,

$$x^2 + y^2 = 1$$

Therefore

$$(\cos\theta)^2 + (\sin\theta)^2 = 1$$

Unit Circle

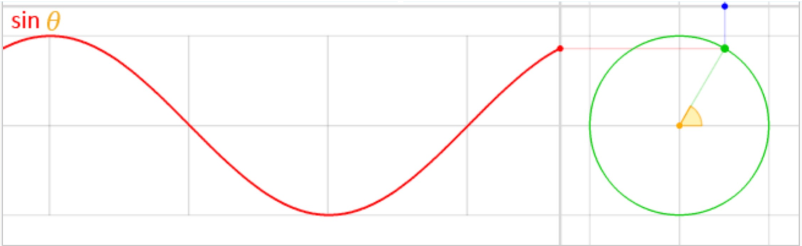


Now let us denote the angle by t.

This is to suggest time.

How do x and y vary as the time passes?

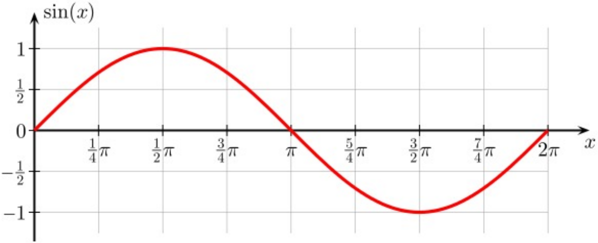
Animation of a Sine Wave

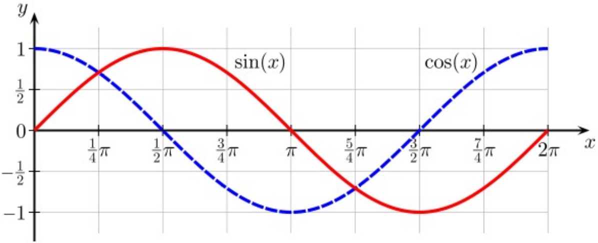


Animation showing how the sine function (in red) $y = \sin(\theta)$ is graphed from the y-coordinate (red dot) of a point on the unit circle (in green) at an angle of θ in radians.

`https://en.wikipedia.org/wiki/Sine/`

Sine Waves over One Period



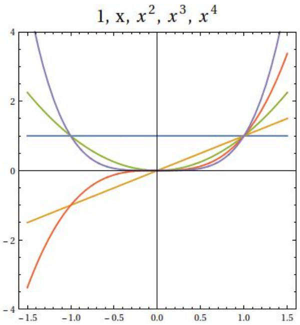


Outline

Introduce Polynomials on BB

Outline the properties and graphs of simple polynomials on the blackboard.

Basis Functions for Approximation



Many functions can be approximated by a series of polynomial functions.

Here we plot the functions

$$1 \quad x \quad x^2 \quad x^3 \quad x^4$$

used as basis functions.

Most functions $f(x)$ can be approximated by a simple polynomial function of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

Polynomial Approximation. Taylor Series

Any "reasonable function" $f(x)$ can usually be approximated by a simple polynomial function

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

Sometimes we can find the roots of the polynomial; that is, the values of x for which it is zero.

Then we are able to write the polynomial as

$$p(x) = a_n (x - x_1)(x - x_2)(x - x_3) \cdots (x - x_n)$$

It is simple to sketch the graph of this function.

Taylor Series for Sine Wave

The Taylor series for $\sin x$ **is**

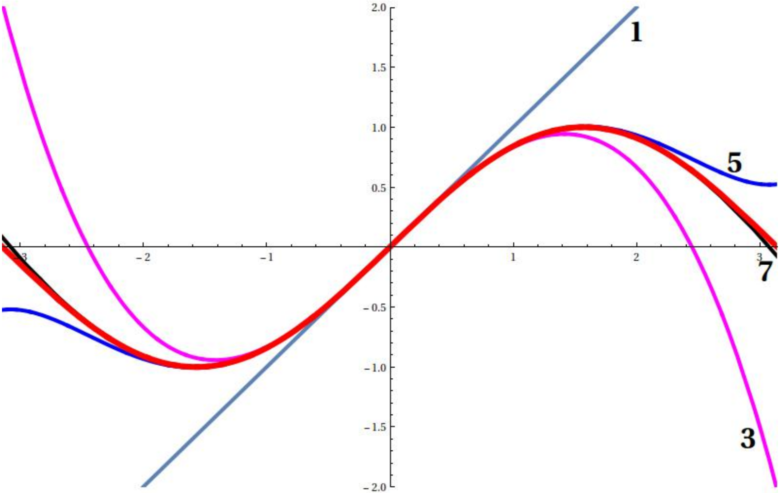
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

We truncate to get a sequence of polynomials:

$$\begin{aligned}
p_1(x) &= x \\
p_3(x) &= x - \frac{x^3}{3!} \\
p_5(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} \\
p_7(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}
\end{aligned}$$

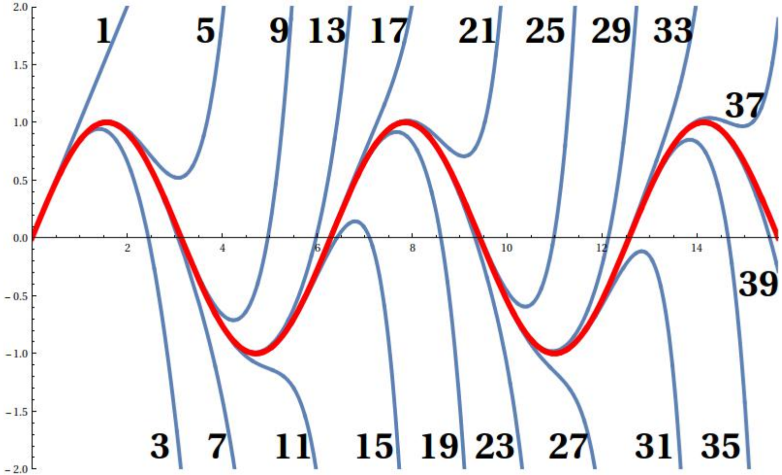
They approximate $\sin x$ **better with increasing order.**

Polynomial Approximation to Sine Wave



$p_7(x)$ **is a good fit over a full wavelength.**

Polynomial Approximation to Sine Wave



$p_{39}(x)$ **fits well over five wavelengths.**

Thank you