AweSums:

The Majesty of Mathematics

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Outline

Outline

AweSums: The Majesty of Maths

Bernhard Riemann (1826-66)

AweSums: The Majesty of Maths

We aim to get a flavour of the Riemann Hypothesis.

It involves the zeros of the "Zeta function":

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

So, we need to talk about several *new topics:*

- **What is a function?**
- **What is an infinite series?**
- **What is a complex variable?**

In this lecture, we will look at infinite series.

A Little Puzzle: Solution

Which is bigger: A Googol or 100!

$$1 \text{ googol} = 10^{100} \qquad 100! = 1 \times 2 \times 3 \times \cdots \times 100$$

Let us follow the example of Gauss:

$$\begin{array}{ccccccccccccc} 100! & = & 1 & \times & 2 & \times & 3 & \times & \cdots & \times & 99 & \times & 100 \\ 1 \text{ googol} & = & 10 & \times & 10 & \times & 10 & \times & \cdots & \times & 10 & \times & 10 \end{array}$$

$$\begin{aligned} 1 \times 100 & & &= 100 \\ 2 \times 99 &= 198 & &> 100 \\ 3 \times 98 &= 294 & &> 100 \\ & \cdots & & \\ 50 \times 51 &= 2550 & &> 100 \end{aligned}$$

So factorial 100 is bigger than a googol. Much bigger!

A Little Puzzle

More technically, we have 50 products of the form

$$[50\tfrac{1}{2} - (n - \tfrac{1}{2})] \times [50\tfrac{1}{2} + (n - \tfrac{1}{2})] \,, \qquad n \in \{1, 2, \ldots, 50\}$$

But this is equal to

$$(50\tfrac{1}{2})^2 - (n - \tfrac{1}{2})^2$$

The smallest value occurs for $n = 50$:

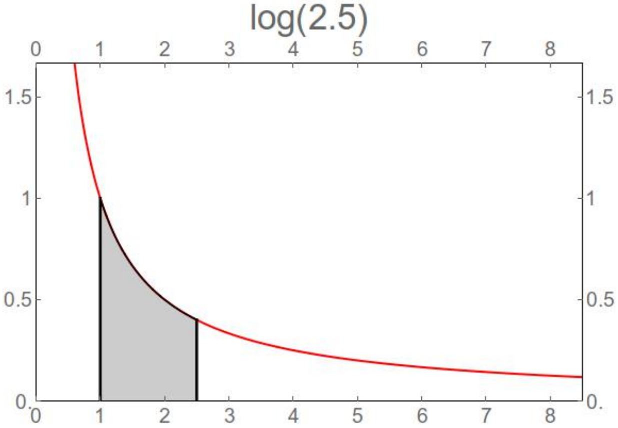
$$(50 + \tfrac{1}{2})^2 - (50 - \tfrac{1}{2})^2 = (50^2 + 50 + \tfrac{1}{4}) - (50^2 - 50 + \tfrac{1}{4}) = 100$$

So all products are greater than or equal to 100.

Outline

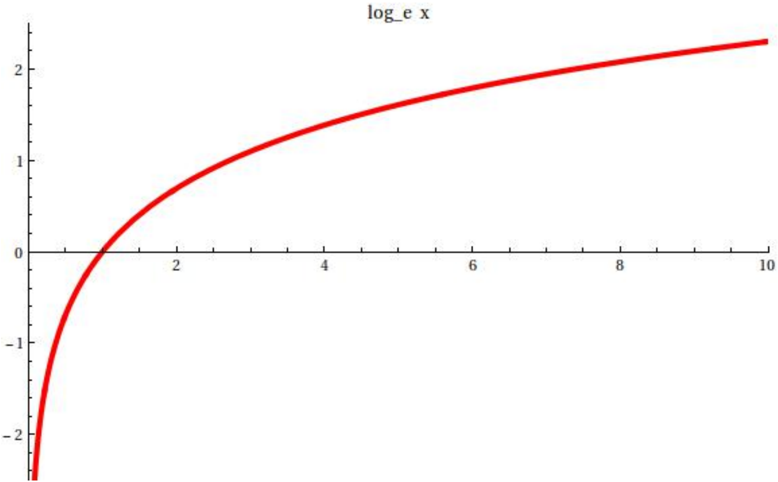
Definition of Natural Logarithm

The natural log is the area shown in this graph:



For example, *log 2.5* is the area is between 1 and 2.5.

$\text{Log}_e\ x$ for $0 < x < 10$



Key Properties

We have found that

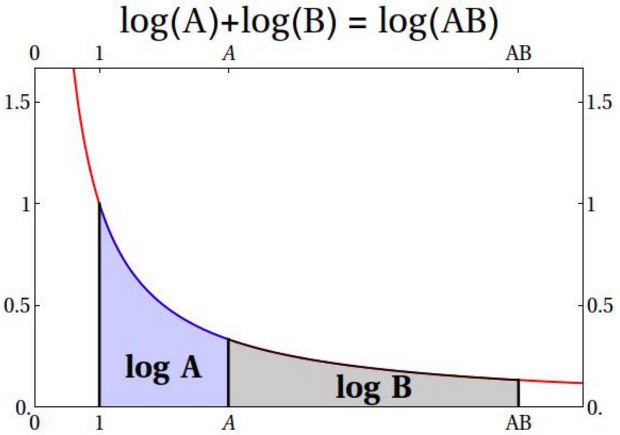
$$
\begin{array}{rcl}
\textbf{For } x > 1 & & \log x > 0 \\
\textbf{For } x = 1 & & \log x = 0 \\
\textbf{For } 0 < x < 1 & & \log x < 0 \\
\log A + \log B & = & \log A\,B
\end{array}
$$

More properties:

$$
\begin{array}{rcl}
\log A - \log B & = & \log A/B \\
\log 1/A & = & -\log A \\
\log A^2 & = & 2\log A \\
\log A^r & = & r\log A
\end{array}
$$

Crucial Property of Logs

We found the important property of logarithms:



It turns multiplication into addition.

Crucial Property of Logs

$$\log A + \log B = \log A\,B$$

Suppose we wish to multiply A by B.

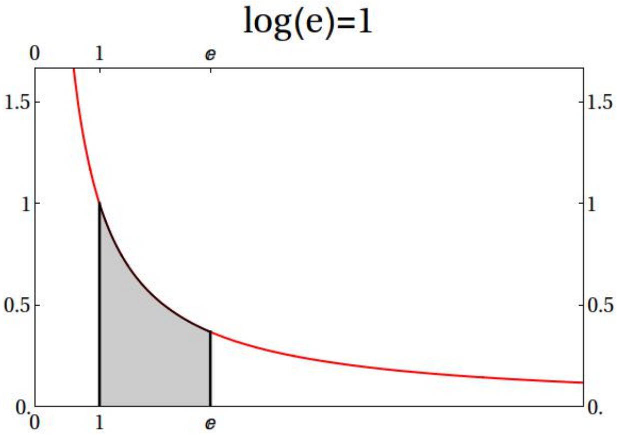
We add the logarithms, log A and log B.

This gives us log AB.

Then we invert the logarithm to get A B.

What Number has Natural Logarithm 1?

There is a number that makes the area equal to one:



This is Euler's number *e*.

Euler's Number e

Euler's number e may be defined in many ways.
For example, it may be defined as a limit:

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

It is often described in terms of compound interest.

Suppose we invest 1 Euro at interest rate X%.
We write $x = \mathbf{X}/100$. After one year, we have:

$$\left(1+\frac{\mathbf{X}}{100}\right) = (1+x) \quad \textbf{Euros}$$

after a year.

If the interest is calculated *every six months*, then two payments are made in a year so we get

$$\left(1+\frac{x}{2}\right)^{2}$$

If the interest is calculated *every three months*, then four payments are made in a year so we get

$$\left(1+\frac{x}{4}\right)^{4}$$

If the interest is calculated n *times per year*, then n payments are made and we get

$$\left(1+\frac{x}{n}\right)^{n}$$

Ultimately, with continuous computation of interest,

$$e^{x}=\lim_{n\longrightarrow\infty}\left(1+\frac{x}{n}\right)^{n}$$

Euler's Number e

Let's calculate this for a few values of n:

$$\begin{array}{rl} \textbf{For } n = 1 & (1+1/1)^1 = 2.0 \\ \textbf{For } n = 2 & (1+1/2)^2 = 2.25 \\ \textbf{For } n = 3 & (1+1/3)^3 = 2.37 \\ \textbf{For } n = 4 & (1+1/4)^4 = 2.44 \\ & \dots \\ \textbf{For } n = 100 & (1+1/100)^{100} = 2.705 \\ \textbf{For } n = 1000 & (1+1/1000)^{1000} = 2.717 \end{array}$$

Euler's Number e

We find that

For $n = 1,000,000$ $\qquad \left(1 + \frac{1}{10^6}\right)^{10^6} = 2.71828046932\ldots$

Continuing to ever-bigger values of n**, we get**

$$e = 2.718281828459045235 4\cdots$$

This is the base of the natural logarithms.

It is generally called Euler's number.

Outline

Exponential Growth & Decay

The exponential function is written

$$y = \exp(x) \qquad \textbf{or} \qquad y = e^x$$

It is the inverse function of the logarithm:

$$y = \exp(x) \qquad \Longleftrightarrow \qquad x = \log(y)$$

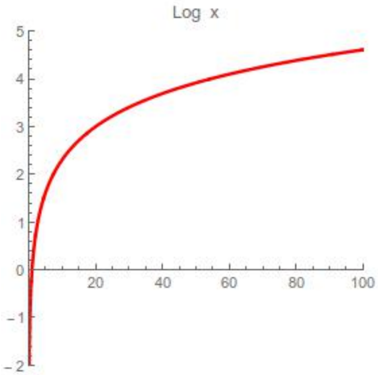
Therefore we have:

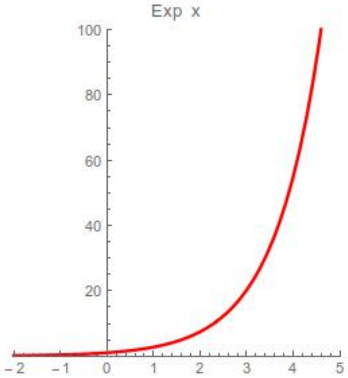
$$y = \exp(\log(y)) \qquad \textbf{and} \qquad x = \log(\exp(x))$$

or

$$y = e^{\log(y)} \qquad \textbf{and} \qquad x = \log(e^x)$$

We can get the graph of $\exp x$ from that of $\ln x$ by rotating about the line $x = y$.





Exponential Growth

Exponential Growth is very common in nature. If the rate of growth is proportional to the size of the population, the growth is exponential.

For example, in a *larger human population,* with more potential for growth, the rate of increase is greater.

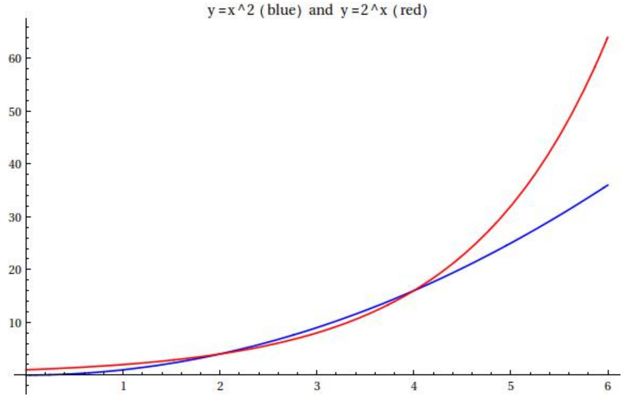
A *bacteria colony* growing exponentially can increase explosively and catastrophically within a few days.

Polynomial and Exponential Growth

Let's compare two functions, $f(n) = n^2$ **and** $g(n) = 2^n$**:**

$$
\begin{array}{lclllllll}
n & = & 0 & 1 & 2 & 3 & 4 & 5 & \cdots & 10 \\
n^2 & = & 0 & 1 & 4 & 9 & 16 & 25 & \cdots & 100 \\
2^n & = & 1 & 2 & 4 & 8 & 16 & 32 & \cdots & 1024
\end{array}
$$

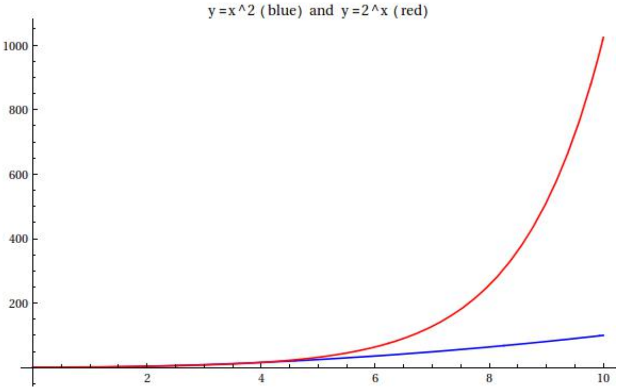
Polynomial *versus* Exponential Growth



At $x = 1$, $2^x = 1$ and $x^2 = 0$. At $x = 2$ they are equal.
Between $x = 2$ and $x = 4$, the exponential is smaller than the quadratic.

Above $x = 4$, the exponential soars into the stratosphere.

Polynomial *versus* Exponential Growth



As x becomes larger, the exponential function $y = 2^x$ becomes completely dominant. Even when $x = 10$, it is an order of magnitude greater than $y = x^2$.

Grababundel and the Maharaja

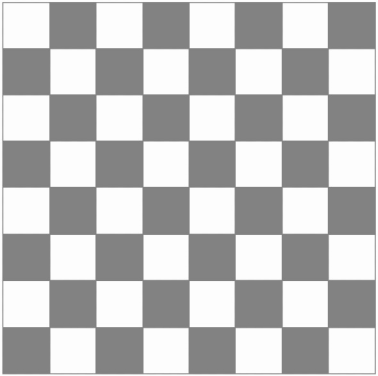
Long ago in the Gupta Empire, a great-but-greedy mathematician, Grababundel, presented to the Maharaja a new game that he had devised.

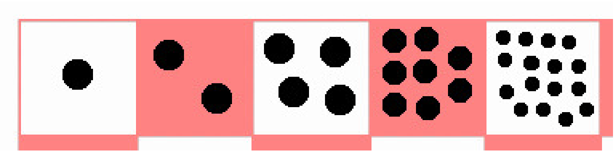
He called it Chaturanga. *We call it Chess.*

The Maharaja, Branier Thanilux, was so pleased that he asked Grababundel to name his reward.

Grabundel said simply: *"Give me one grain of rice for the first square on the board, two for the second, four for the third, and so on, doubling each time, until the 64th square. That is all I ask!"*

The Great-but-Greedy Mathematician Grababundel and his Chessboard





The number of grains of rice comes to

$$N = 1 + 2 + 4 + 8 + \cdots + 2^{63}$$

How many is that?

It is a geometric series. It can be summed.

But there is crafty and simple way to get N:

Just add 1 to the sum and *watch the cascade.*

(computation on blackboard).

The Maharaja Outsmarts Greedy Grababundel

Branier Thanilux saw through the ruse. He said to Grababundel: “You are too modest; if you wish, I will give you a boundless fortune, riches without limit."

For the thrilling story of how the Maharaja got the better of Grabundel, read my blog-post “Chess Harmony” at `thatsmaths.com`.

The moral of the story:

Don’t mess with the Maharaja —
He might be Branier Thanilux.

Outline

Leonhard Euler (1707–1783)

Leonhard Euler was the most prolific mathematician and one of the greatest of all time.

Contributed to every area of mathematics, both pure and applied.

His collected works fill some 80 volumes.

Outline of Euler's Life

Born in 1707 in Basel, Euler studied for a time under Johann Bernoulli.

Took up a position in *St. Peterburg* in 1727, at the Academy established by Peter the Great.

Married Katharina Gsell in 1734. Of their thirteen children, just five survived beyond infancy.

Outline of Euler’s Life

In 1741, Euler moved to the Academy in Berlin, where Frederick the Great of Prussia had offered him a position. Euler stayed 25 years in Berlin.

Catherine the Great came to power in 1762.
In 1766, Euler returned to St. Petersburg.

Around that time he lost his sight almost completely. However, his mathematical output did not diminish; in fact, he became even more productive!

Euler remained in Russia until his death in 1783.

Russian stamp for 250th birthday

Notation invented or popularised by Euler

$$e \qquad i \qquad \pi \qquad f(x) \qquad \sum$$

$$\sin \quad \cos \quad \tan \quad \csc \quad \sec \quad \cot$$

Mathematical formula voted the most beautiful:

$$e^{i\pi} + 1 = 0$$

Some Key Accomplishments

Euler's mathematical accomplichments were profound and frequently breath-taking.

We will focus here on just two results:

1. The Basel Problem
2. The Product-Sum Formula

The Basel Problem

Recall the definition of Riemann's "Zeta function":

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

For $s = 2$ this is the series

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \dots$$

In 1734, Euler found the precise value of this series:

$$\zeta(2) = \frac{\pi^2}{6}$$

This result brought him great fame.

The Product-Sum Formula

Leonhard Euler proved the amazing result:

$$\zeta(s) = \sum_{n\in\mathbb{N}} \frac{1}{n^s} = \prod_{p\in\mathbb{P}} \left(\frac{1}{1-p^{-s}} \right)$$

This was an utterly unexpected result:

It connects $\zeta(s)$ with the prime numbers.

It was the beginning of analytical number theory, and is central to the Riemann hypothesis.

An Aweful Limerick by W. C. Willig

Exp$(i\pi) + 1 = 0$
Made the great Leonhard Euler a hero
From real to complex
With our brains in great flex
He led us with zest but no fearo!

[Credited to W. C. Willig]

Challenge:

Write a better one. You can hardly do worse!

An Ode to *e*

To inspire you, here is an ode, to be sung
to the air of *"Only God can make a tree"*

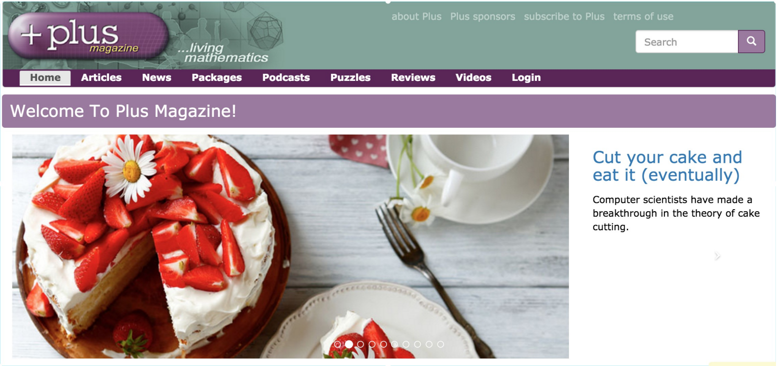
I think that I shall never see
A number lovelier than e,
Whose digits are too great to state
They're 2.71828 . . .
. . .

[Credited to Arthur Benjamin]

SOURCES:

- Wikipedia page on Leonhard Euler
- MacTutor:
  `http://www-history.mcs.st-and.ac.uk/`
  (J. J. O'Connor & E. F. Robertson)
- `eulerarchive.maa.org`
- William Dunham's book *Journey through Genius.*

Distraction 7: Plus Magazine



PLUS: The Mathematics e-zine

`https://plus.maths.org/`

Outline

Sequences & Series

A *sequence* is a set of numbers, $s_1, s_2, s_3, \cdots$ indexed by the natural numbers:

$$S = \{s_1, s_2, s_3, \ldots\ s_n, \ldots\}$$

For example, the sequence of squares is

$$\begin{aligned} S &= \{1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2 \ldots\} \\ &= \{1, 4, 9, 16, 25, 36, 49 \ldots\} \end{aligned}$$

Another example is the sequence of prime numbers

$$S = \{2, 3, 5, 7, 11, 13, 17, 19, 23 \ldots\}$$

Infinite Series

A *series* is a sum of numbers $a_1 + a_2 + a_3 + \ldots$ indexed by the natural numbers:

$$A = a_1 + a_2 + a_3 + \cdots + a_n + \ldots$$

For example, the sum of the natural numbers

$$A = 1 + 2 + 3 + 4 + 5 + 6 + \ldots .$$

This clearly gets bigger without limit: $A \to \infty$. How do we handle this?

We consider the *sequence of partial sums*:

$$s_1 = a_1 , \ s_2 = a_1 + a_2 , \ s_3 = a_1 + a_2 + a_3 , \ \ldots$$

Again, the *sequence of partial sums* of the series

$$A = a_1 + a_2 + a_3 + \cdots + a_n + \dots$$

is

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ s_4 &= a_1 + a_2 + a_3 + a_4 \\ s_5 &= a_1 + a_2 + a_3 + a_4 + a_5 \\ &\dots \end{aligned}$$

If the sequence $\{s_n\}$ tends to a limit,
we say the series A is *convergent*.

Otherwise, we say the series is *divergent*.

For the series of natural numbers

$$A = 1 + 2 + 3 + 4 + 5 + 6 + \dots .$$

the sequence of partial sums is

$$\begin{aligned}
s_1 &= & 1 & = 1 \\
s_2 &= & 1 + 2 & = 3 \\
s_3 &= & 1 + 2 + 3 & = 6 \\
s_4 &= & 1 + 2 + 3 + 4 & = 10 \\
s_5 &= & 1 + 2 + 3 + 4 + 5 & = 15 \\
& & \dots &
\end{aligned}$$

Clearly, this sequence does not converge.
We say that it *diverges:* $A \to \infty$. No Surprise!

Convergence & Divergence

Definition: A series *converges* if its sequence of partial sums converges.

Definition: A series *diverges* if its sequence of partial sums diverges.

But when does a sequence converge?

A sequence *converges* to S if its terms get closer and closer to S.

This is hardly a rigorous definition.

The true definition is an ϵ-δ definition. We will not give that definition here.

A Geometric Series

We look at the series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

where each term is half the previous one.

The partial sums of this series are

$$\begin{aligned} s_1 &= & 1 & = 1 \\ s_2 &= & 1 + \tfrac{1}{2} & = 1\tfrac{1}{2} \\ s_3 &= & 1 + \tfrac{1}{2} + \tfrac{1}{4} & = 1\tfrac{3}{4} \\ s_4 &= & 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} & = 1\tfrac{7}{8} \\ & & \dots & \end{aligned}$$

This sequence appears to be convergent.

Convergence of the Geometric Series

We look at the partial sum

$$s_n = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots + \tfrac{1}{2^{n-1}} .$$

$$\tfrac{1}{2} s_n = \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots + \tfrac{1}{2^{n-1}} + \tfrac{1}{2^n} .$$

Subtract the second equation from the first:

$$s_n - \tfrac{1}{2} s_n \quad = \quad \tfrac{1}{2} s_n \quad = \quad 1 - \tfrac{1}{2^n} .$$

As n gets larger , this gets closer to 1:

$$\tfrac{1}{2} s_n \approx 1 \qquad \text{or} \qquad s_n \approx 2 .$$

We say that the limit of s_n as $n \to \infty$ is 2:

$$\lim_{n\to\infty} s_n = 2$$

Convergence of the Geometric Series

Again, s_n gets closer and closer to 2 as n becomes larger and larger.

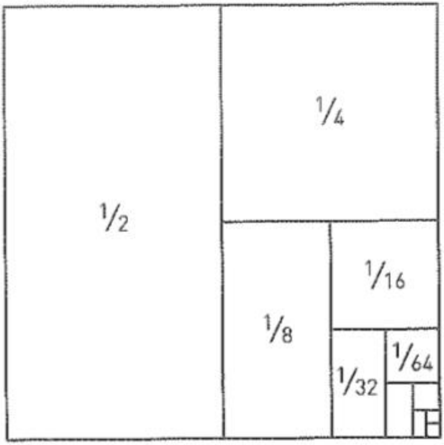
We conclude that the sum of the series is 2:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \longrightarrow 2$$

The geometric series is convergent.

Let us demonstrate this with a picture.

We divide a *unit square* into ever-smaller rectangles:



$$\textbf{Area} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} \cdots = 1$$

Convergence in General

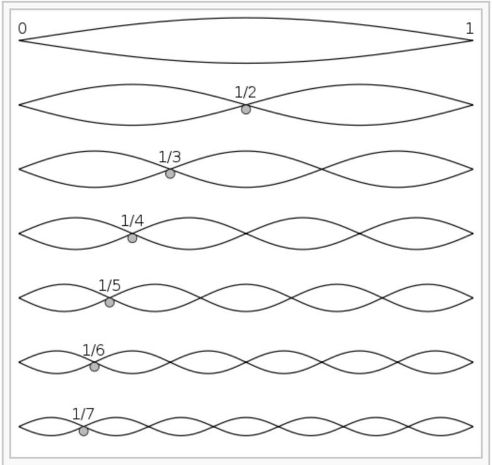
If the terms of a series do not tend to zero, it cannot possible converge: the partial sums are not getting closer to each other.

If the terms of a series do tend to zero, as with the geometric series, there is a chance that it may converge. <u>But this is not guaranteed.</u>

We now consider a series for which the terms become smaller without limit, but which does not converge.

We examine the *Harmonic Series.*

Harmonic Numbers & Musical Harmony



The connection with music goes back to Pythagoras.

The Harmonic Series

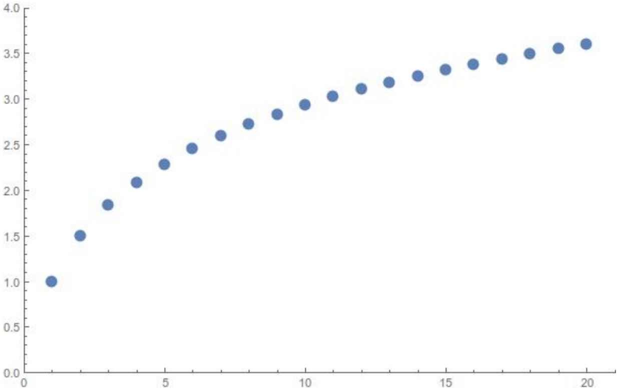
We define the harmonic series as

$$H = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

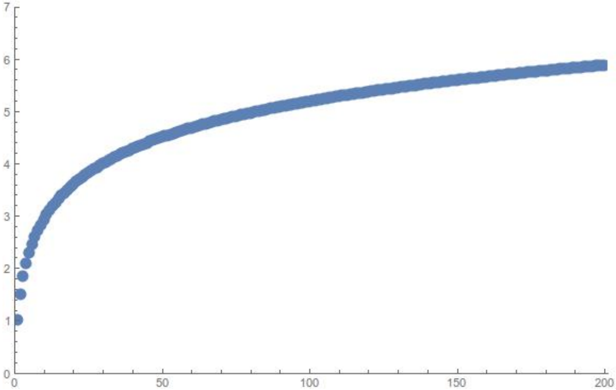
The partial sums are called the harmonic numbers

$$\begin{aligned}
H_1 &= & 1 & = 1 \\
H_2 &= & 1 + \tfrac{1}{2} & = 1\tfrac{1}{2} \\
H_3 &= & 1 + \tfrac{1}{2} + \tfrac{1}{3} & = 1\tfrac{5}{6} \\
H_4 &= & 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} & = 2\tfrac{1}{12} \\
H_5 &= & 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} & = 2\tfrac{17}{60} \\
& & \cdots &
\end{aligned}$$

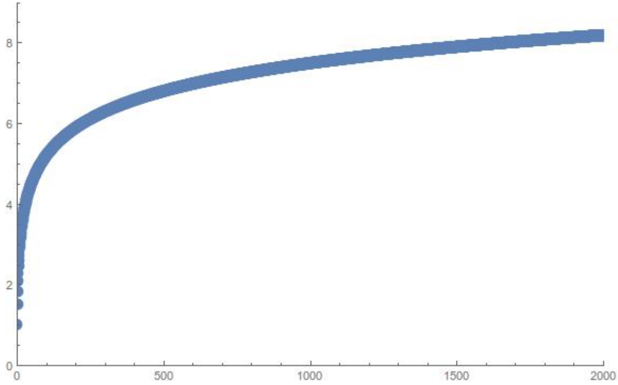
The sums are getting bigger. What will happen?



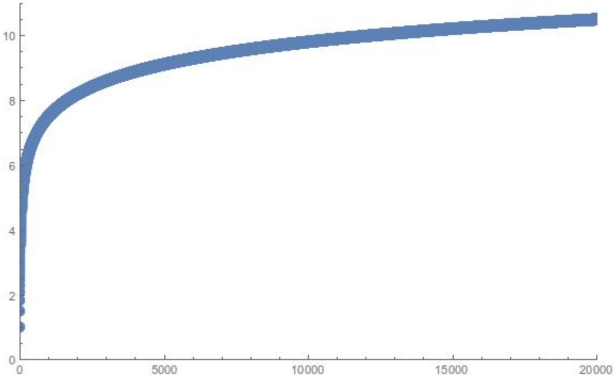
Harmonic numbers from 1 to 20



Harmonic numbers from 1 to 200

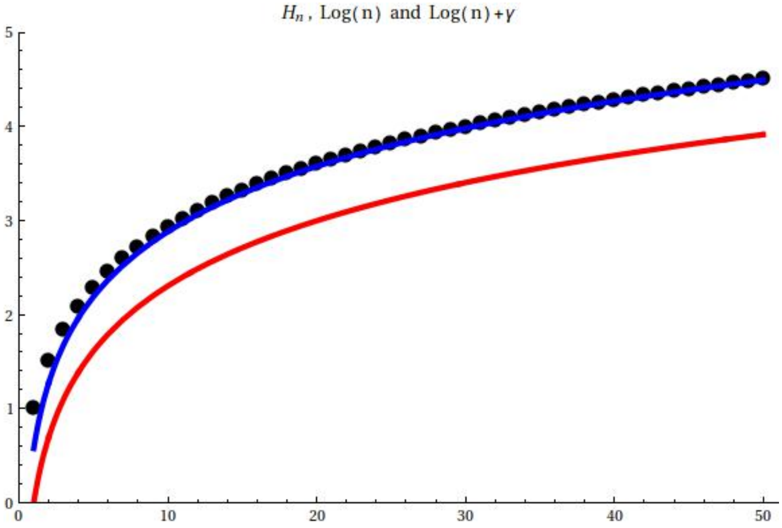


Harmonic numbers from 1 to 2000



Harmonic numbers from 1 to 20000

Harmonic Numbers and Logs



Divergence of the Harmonic Series

We rearrange the terms of the harmonic series

$$H = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

into groups of terms.

$$\begin{aligned}
H = \quad & \left(\tfrac{1}{1}\right) && = 1 \\
& + \left(\tfrac{1}{2}\right) && = \tfrac{1}{2} \\
& + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) && > \tfrac{1}{2} \\
& + \left(\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}\right) && > \tfrac{1}{2} \\
& + \left(\tfrac{1}{9} + \tfrac{1}{10} + \tfrac{1}{11} + \tfrac{1}{12} + \tfrac{1}{13} + \tfrac{1}{14} + \tfrac{1}{15} + \tfrac{1}{16}\right) && > \tfrac{1}{2} \\
& \cdots
\end{aligned}$$

Each group is equal to or greater than $\frac{1}{2}$.
The sum becomes larger without limit: it diverges!

Convergence & Divergence of Series

The geometric series converges

$$G = \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 2$$

The harmonic series diverges

$$H = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \infty$$

For both series, the terms get smaller and smaller and tend towards zero.

Clearly, this is not enough to ensure convergence.

We have

$$\begin{bmatrix}\text{Series}\\ \text{converges}\end{bmatrix} \Rightarrow \begin{bmatrix}\text{Terms become}\\ \text{smaller}\end{bmatrix}$$

We do not have

$$\begin{bmatrix}\text{Terms become}\\ \text{smaller}\end{bmatrix} \Rightarrow \begin{bmatrix}\text{Series}\\ \text{converges}\end{bmatrix}$$

and certainly not

$$\begin{bmatrix}\text{Terms become}\\ \text{smaller}\end{bmatrix} \Leftrightarrow \begin{bmatrix}\text{Series}\\ \text{converges}\end{bmatrix}$$

Splitting up the Harmonic Series

We write alternate terms as two separate series:

$$H_{\mathrm{ODD}} = \frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \cdots$$

$$H_{\mathrm{EVEN}} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \cdots$$

Question: Do these series converge or diverge?
Does one converge and one diverge?

Take a few minutes to develop an argument.

Talk to the person beside you if you like;

Splitting up the Harmonic Series

$$
\begin{aligned}
2 \times H_{\mathrm{EVEN}} &= 2 \times \left(\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{6} + \tfrac{1}{8} + \tfrac{1}{10} + \cdots\right) \\
&= \tfrac{1}{1} + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \cdots \\
&= H
\end{aligned}
$$

So the sum of even terms diverges since H diverges.

Since

$$H_{\mathrm{EVEN}} = \tfrac{1}{2} H \qquad \text{and} \qquad H = H_{\mathrm{ODD}} + H_{\mathrm{EVEN}}$$

we could argue that $H_{\mathrm{ODD}} = \frac{1}{2} H$.

It is better to use a 'dominance argument' for H_{ODD}.

The Alternating Harmonic Series

We define the alternating harmonic series:

$$A = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \cdots$$

This is a modified form of the harmonic series in which the signs of the terms alternate.

Does this series converge? Yes!

"It can be shown" that $A = \log 2$:

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \cdots = \log 2$$

Riemann's Rearrangement Theorem

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \log 2$$

We can *rearrange the terms* to get *any value we like.*

Weird! This follows from a theorem of Riemann.

The *algorithm* is simply described:

- **Suppose we wish the sum to be $S = 2\pi$.**
- **Add positive terms until the sum exceeds S**
- **Now add negative terms until it is less than S**
- **Continue this procedure indefinitely.**

The result is that the sum tends towards S.

Puzzle: The Galway Girl

I want to see my girlfriend in Galway.
But I'm shy! Will I ever get there?

- I travel half-way to Galway.
- Losing my nerve, I return towards Dublin.
- But, half-way back, I regain courage.
- I travel half the distance to Galway.
- Then I travel half the distance to Dublin.
- Back and forth, hither and thither . . .

Is there any hope or will my love remain unrequited?

Thank you