AweSums:

The Majesty of Mathematics

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Outline

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AweSums: The Majesty of Maths

Bernhard Riemann (1826-66)

AweSums: The Majesty of Maths

We aim to get a flavour of the Riemann Hypothesis.

It involves the zeros of the "Zeta function":

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

So, we need to talk about several *new topics:*

- **What is a function?**
- **What is an infinite series?**
- **What is a complex variable?**

In this lecture, we will look at functions.

Outline

A First Look at Functions

The concept of a function is amongst the most fundamental and important ideas in mathematics.

A function is a relation between input values and and output values.

Functions are of central importance because they describe connections between sets.

For each input, there is precisely one output.

Notation for Functions

We use the following notation

- x is the input
- y is the output
- f is the function

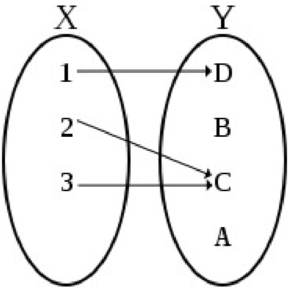
Then we write the function as

$$y = f(x)$$

We call x the independent variable.
We call y the dependent variable.

The set of values taken by x is the domain.
The set of values taken by y is the codomain.

Example of a Function



Domain: $X = \{1, 2, 3\}$

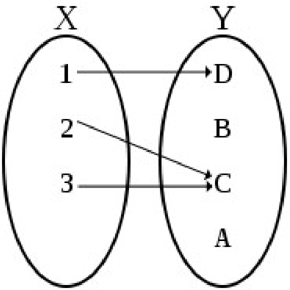
Codomain: $Y = \{A, B, C, D\}$

Range: $\{C, D\}$

Graph: $\{(1, D), (2, C), (3, C)\}$

Example of a Function

X is the domain. Y is the codomain.



D is the image of 1.

1 is the preimage of D.

$\{2, 3\}$ is preimage of C.

A, B have no preimages.

Example of a Function

Square function: Output is square of the input.

We suppose that x can take any real value.

We write this function as

$$y = x^2 \,, \qquad x \in \mathbb{R}$$

Note that different inputs may give the same output:

$$3^2 = 9 \qquad \textbf{and} \qquad (-3)^2 = 9$$

So, in general, a function is not a one-to-one correspondence.

Specifying Functions

A function may be defined in several ways:

- As a Table of Values
- As a Formula
- As a Graph
- As an Algorithm
- As a Solution of an Equation
- Implicitly (e.g. inverse function)

Function Defined by a Table

Input: MONTH
Output: RAINFALL

Table : Average Monthly Rainfall in Dublin

January	78 mm
February	76 mm
March	69 mm
...	...
December	72 mm

Annual precipitation in Dublin: 750 mm.

Function Defined by a Formula

We have already seen the square function:

$$y = x^2 \,, \qquad x \in \mathbb{R}$$

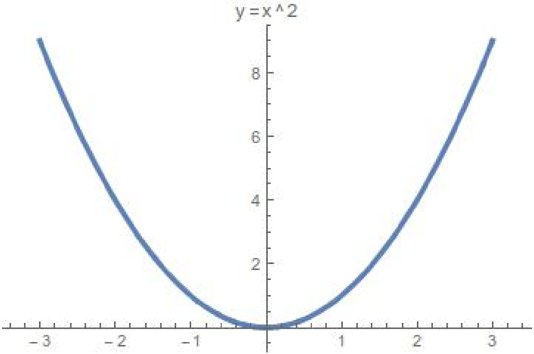
Here are some others:

$$\begin{aligned}
y(x) &= 4x + 6 \\
y(x) &= ax^2 + bx + c \\
y(x) &= (x^2 + 5)/(3x^3 + 7) \\
y(x) &= A \sin \alpha x + B \cos \beta x \\
\zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s} \\
\Gamma(s) &= \int_0^{\infty} e^{-x} x^{s-1} \, \mathrm{d}x
\end{aligned}$$

Function Defined by a Graph

The set of all (input, output) pairs is called the *graph:*

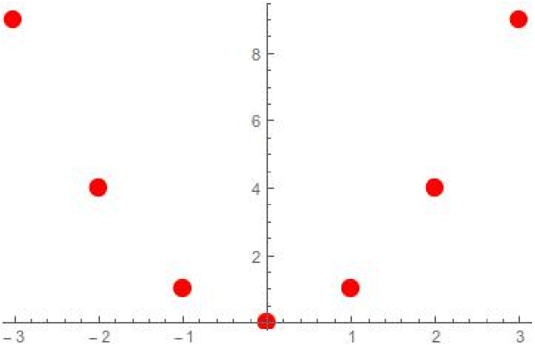
$$G = \{(x, x^2) : x \in [-3, +3]\}$$



Function of a Discrete Variable

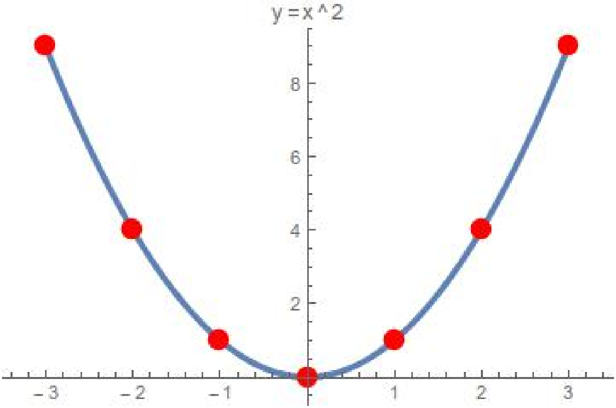
We may restrict the definition to a discrete domain:

$$G = \{(n, n^2) : n \in \{-3, -2, -1, 0, 1, 2, 3\}\}$$

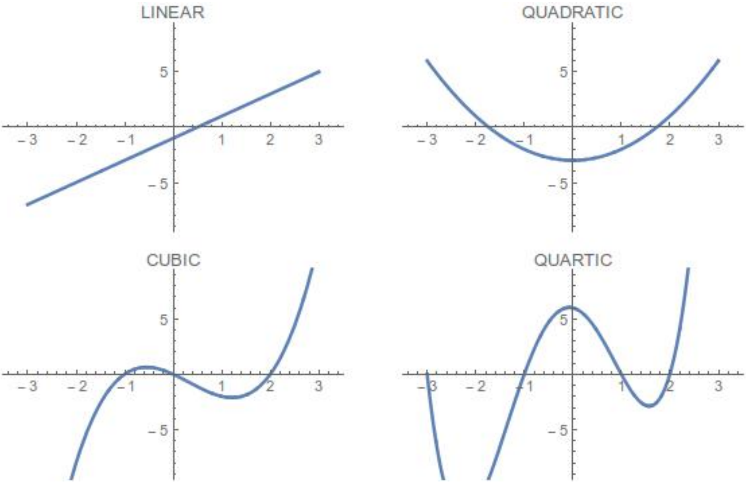


Discrete & Continuous Domains

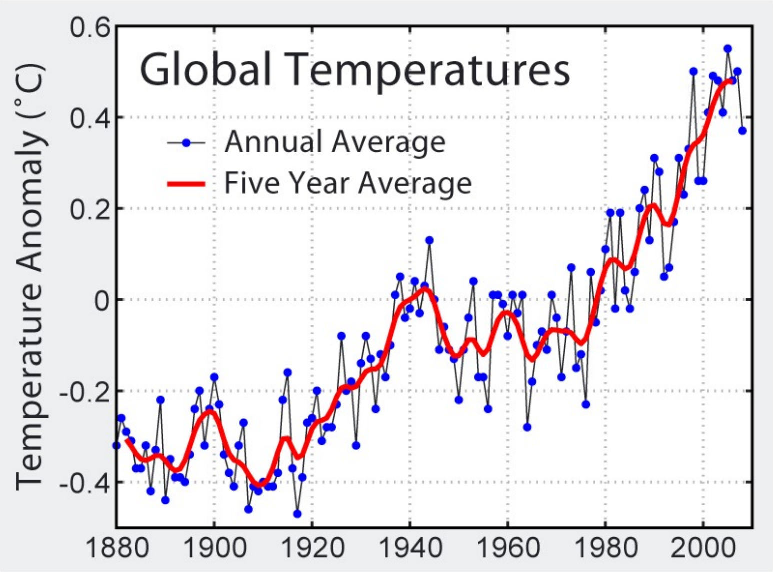
Plot of discrete and continuous functions together:



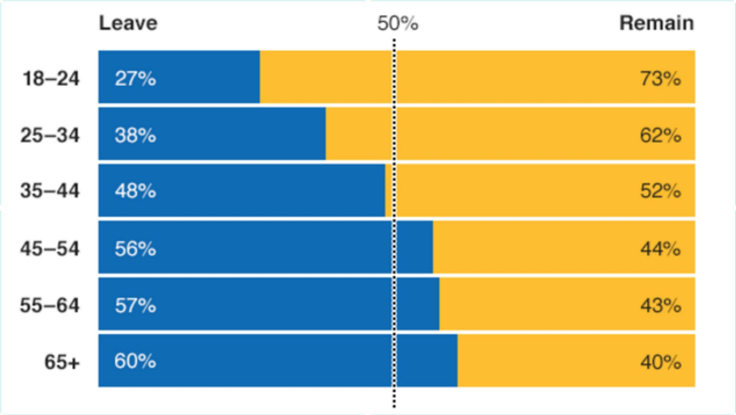
Polynomial Function Graphs



Global Mean Surface Temperature

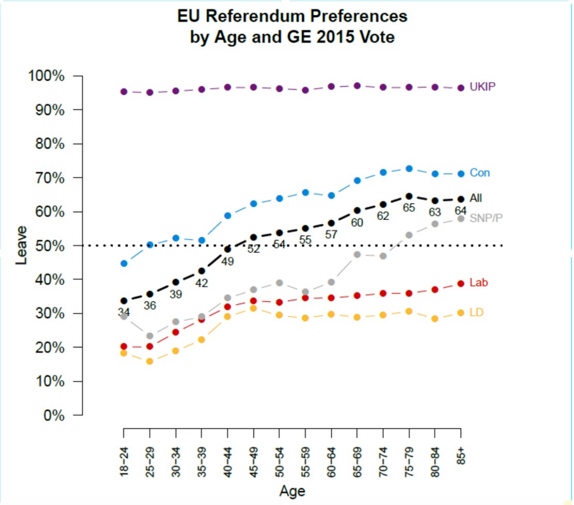


Graphs of Brexit Results: June 2016



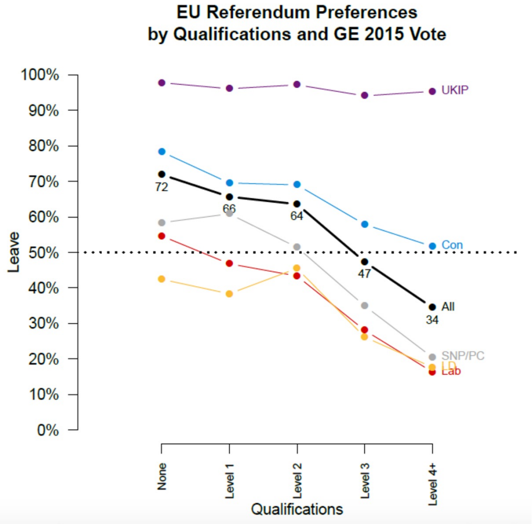
Question: What is the independent variable here?

Graphs of Brexit Results: June 2016



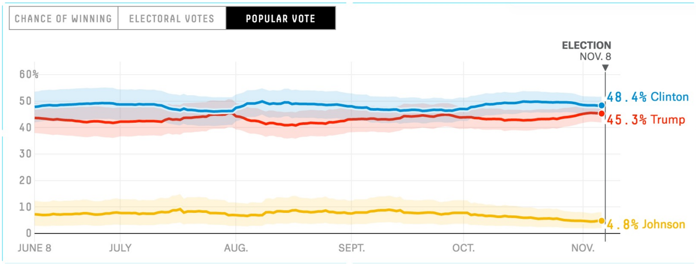
Voting by age group and party affiliation.

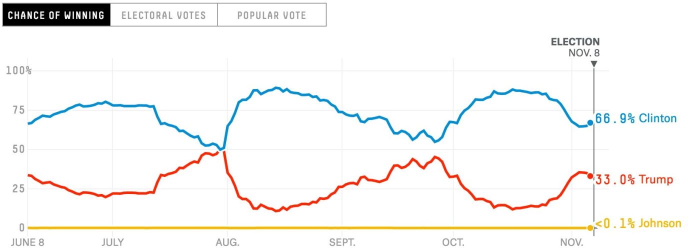
Graphs of Brexit Results: June 2016



Voting by educational level.

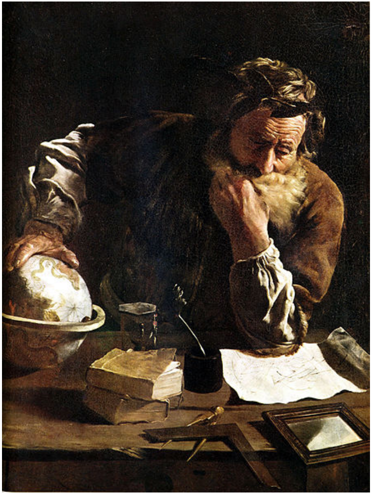
American Presidential Election Trends





Outline

$A\rho\chi\iota\mu\eta\delta\eta\varsigma$



Archimedes Thoughtful **by Domenico Fetti (1620)**

Archimedes of Syracuse (287-212)

Archimedes was a brilliant physicist, engineer and astronomer, the greatest mathematician of antiquity.

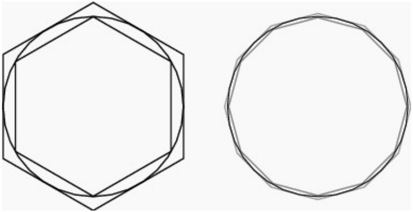
He is famed for:

- **Founding hydrostatics**
- **Formulating the law of the lever**
- **Inventing a helical pump**
- **Designing engines of war**
- **Many more things.**

But his mathematical discoveries were his greatest achievements.

Estimation of π

Archimedes determined π by considering polygons inscribed within a circle and polygons around it.



A regular hexagon within a unit circle has length 3.

This is less than the circumference of the circle.
So π is greater than 3.

A less obvious derivation shows that a hexagon drawn around the circle has length $2\sqrt{3}$.

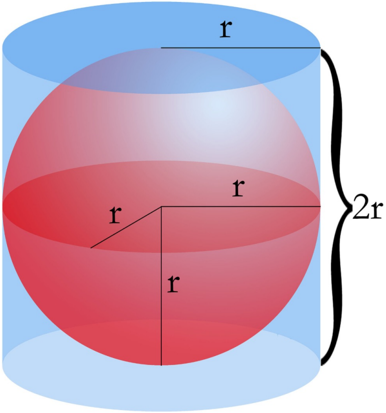
So π is less than $2\sqrt{3} \approx 3.46$. Therefore

$$3 < \pi < 3.46$$

Archimedes approximated the circle by inscribed and circumscribed 96-sided polygons. He found:

$$3\tfrac{10}{71} < \pi < 3\tfrac{10}{70} \qquad \textbf{or} \qquad 3.140845 < \pi < 3.142857$$

Archimedes Great Discovery



Volume of Cylinder:

$$V_C = \pi r^2 \times 2r$$

Volume of Sphere:

$$V_S = \frac{2}{3} V_C$$

Therefore

$$V_S = \frac{4}{3} \pi r^3$$

Cylinder and Sphere

Archimedes showed that a sphere inscribed in a cylinder has two-thirds the volume of the cylinder.

He asked for a sphere within a cylinder
to be inscribed on his tombstone.

Centuries later, the Roman orator Cicero found such a carving on a grave in Syracuse.

[2,300 years later, I did not.]

Distraction 6: Slicing a Pizza

Cut the pizza using three straight cuts.

There should be exactly one piece of pepperoni on each slice of pizza.

Outline

What's the Point of Logs?

What's so interesting about logs?

Why should we bother about them?

We will take a look at some applications of logarithms in 'real-world' situations.

Recall the Definition of Log_{10} x

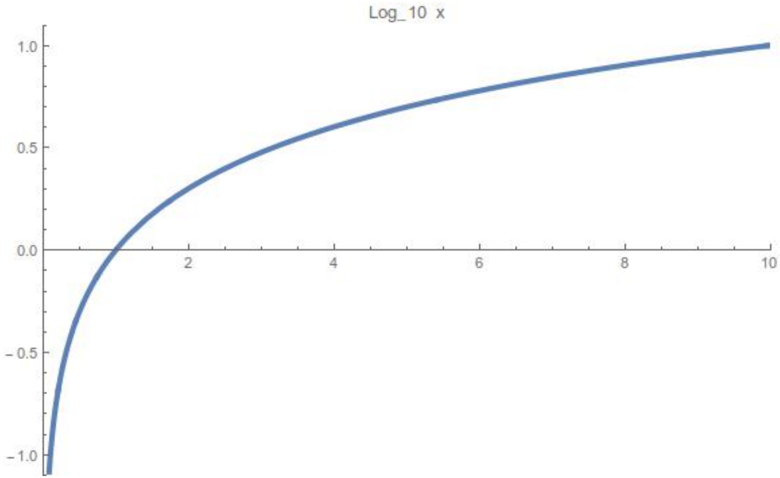
DEFINITION: **The logarithm of x is the power to which 10 must be raised to give x:**

$$\log_{10} y = x \iff 10^x = y$$

Logs compress large values and stretch small ones.

This is clear from the character of the graph.

$\text{Log}_{10}\ x$ for $0 < x < 10$



Our Logarithmic World

Log tables have been consigned to the scrap heap.

But logarithms remain at the core of science.

A wide range of physical phenomena follow logarithmic laws. *We live in a logarithmic world.*

Our Logarithmic World

Many physical variables can take values covering several orders of magnitude.

Smaller values can be swamped or masked by larger values. A log scale compresses them to a more manageable range.

Graphs of quantities that vary exponentially can be converted to linear graphs.

Log Scale of Apparent Magnitude

Log laws are used to model human perception.
Visual perception of brightness is logarithmic.

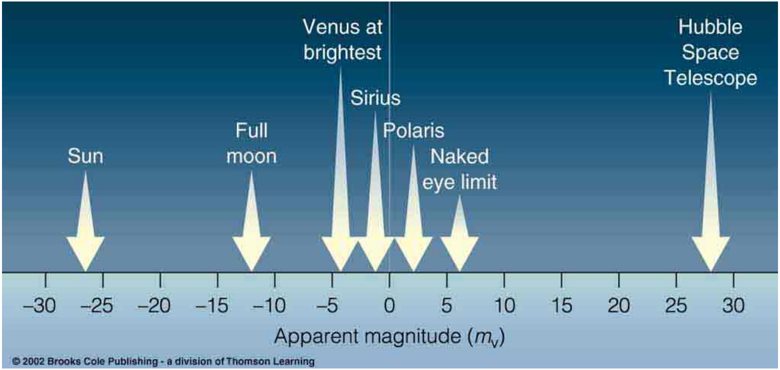
The brightness of stars varies over a huge range.
Astronomers use a log scale for magnitude:

$$m_V = -2.5 \log_{10}(F/F_0)$$

With the star *Vega* as a zero reference, the dimmest star visible to the naked eye is of magnitude 5.

The Full Moon has magnitude -12
and the Sun is of magnitude -26.

Log Scale for Apparent Magnitude



Log Scale in Acoustics

The intensity of audible sounds can vary by a factor of over a trillion (12 orders).

Sound intensity is measured in decibels:

$$L = 10 \log_{10} \left(\frac{P}{P_0} \right)$$

where P_0 is the threshold of hearing.

- **A whisper is 20 dB,**
- **Normal speech is about 60 dB**
- **Rock concert over 110 dB.**

Log Scale for Noise Levels

Source	Pressure rms (Pa)	Sound Intensity level SIL (dB)	Intensity (W/m^2)
Jet engine at 10 m		150	10^3
Jet engine	200	140	100
Jack hammer	60	130	10
Car horn	20	120 (pain threshold)	1
Rock band	6	110	0.1
Machine shop	2	100	0.01
Train	0.6	90	10^{-3}
Vacuum cleaner	0.2	80	10^{-4}
TV	0.06	70	10^{-5}
Conversation	0.02	60	10^{-6}
Office	0.006	50	10^{-7}
Library	0.002	40	10^{-8}
Hospital	0.0006	30	10^{-9}
Broadcast studio	0.0002	20	10^{-10}
Rustle of leaves	0.00006	10	10^{-11}
Threshold of hearing	0.00002	0	10^{-12}

The Richter Scale

The scale named for the American seismologist Charles Richter measures the energy of a quake.

$$R = \log_{10}\left(\frac{A}{A_0}\right)$$

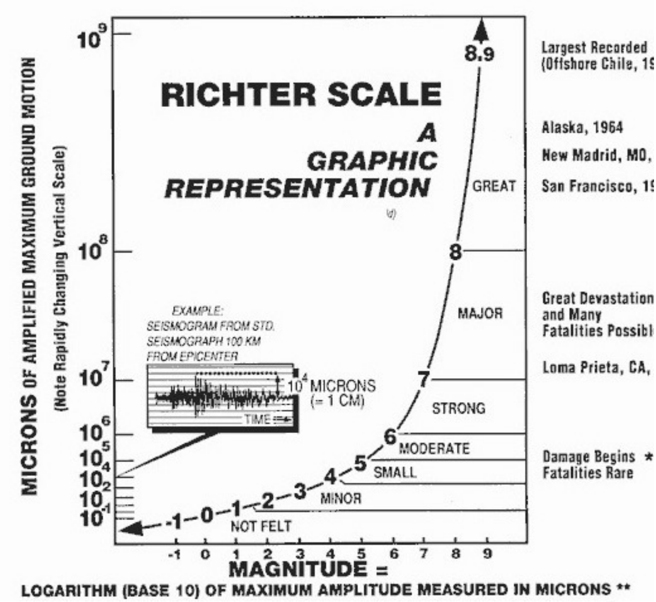
Where A_0 is a barely perceptible tremor.
An increase of 1 implies factor of 10 for amplitude.

A quake of magnitude 6 releases 32 times more *energy* than one of magnitude 5, and a quake of magnitude 7 releases 1000 times more energy.

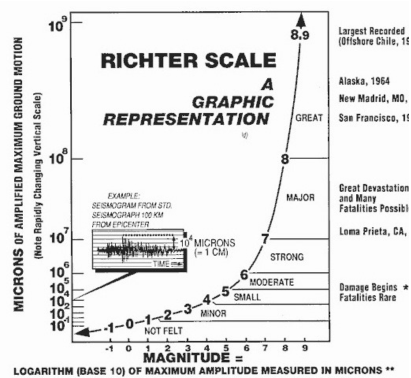
[Empirically, energy goes as 3/2 power of amplitude.

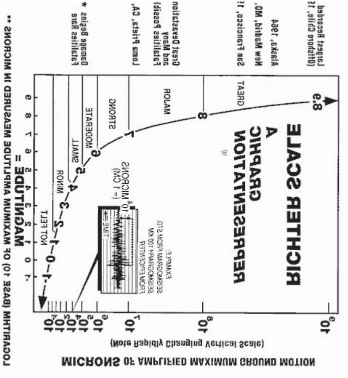
Increase of R by 2 means 1000 times more energy.]

Richter Scale for Earthquakes



Flipping the Richter Scale





The pH Scale of Acidity

The concentration of hydrogen ions in an aqueous solution determines whether it is acidic or alkaline.

Concentration can vary by ten or more orders.

To cover the whole range of possibilities, the acidity of the solution is expressed in logarithmic form.

- **Acidic: pH below 7**
- **Neutral: pH equal to 7**
- **Basic: pH above 7**

Log Scale for Acidity/Alkalinity

Concentration of Hydrogen ions compared to distilled water		Examples
10,000,000	pH 0	Battery acid
1,000,000	pH 1	Hydrochloric acid
100,000	pH 2	Lemon juice, vinegar
10,000	pH 3	Grapefruit, soft drink
1,000	pH 4	Tomato juice, acid rain
100	pH 5	Black coffee
10	pH 6	Urine, saliva
1	pH 7	"Pure" water
1/10	pH 8	Sea water
1/100	pH 9	Baking soda,
1/1,000	pH 10	Great Salt Lake
1/10,000	pH 11	Ammonia solution
1/100,000	pH 12	Soapy water
1/1,000,000	pH 13	Bleach
1/10,000,000	pH 14	Liquid drain cleaner

The Prime Number Theorem

THE REAL REASON WE ARE STUDYING LOGS.

The log function is intimately connected with the distribution of prime numbers.

PNT: The number of primes less than n is

$$\pi(n) \sim \frac{n}{\log_e n}$$

This is intimately connected with the *Riemann Hypothesis.*

A Little Exercise

Note: All logs are to the base 10.

- What is the log of a googol?
- What is log-log of a googol?
- What is the log of a googolplex?
- What is log-log of a googolplex?
- What is log-log-log of a googolplex?
- What is log-log-log-log of a googolplex?

Remember:

$$1 \text{ googol} = 10^{100} \qquad 1 \text{ googolplex} = 10^{\text{googol}}$$

Outline

Definition of Logarithms

Recall how we defined a logarithm:

$$y = \log_b x \quad \Longleftrightarrow \quad x = b^y = b^{\log_b x}$$

Here y is the log of x to the base b.

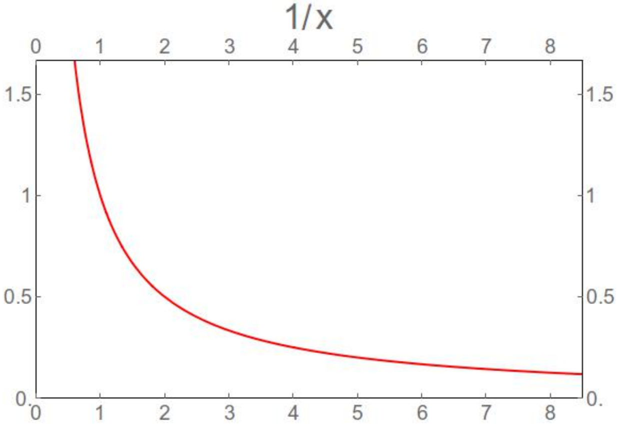
For example, common logs have base 10:

$$10^3 = 1000 \quad \Longrightarrow \quad 3 = \log_{10} 1000 = \log_{10} 10^3$$

We now consider natural logs, having base e.
(We will define e shortly.)

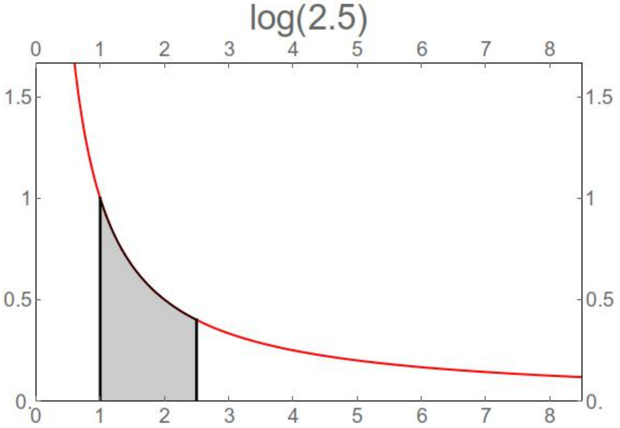
y = 1/x

We will look at the area under the hyperbola $y = 1/x$:



Definition of Natural Logarithm

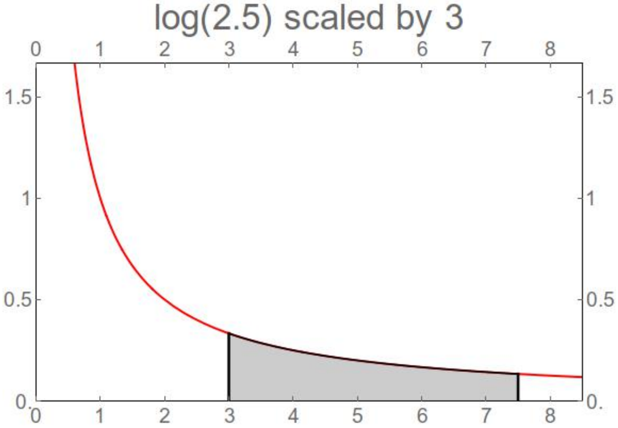
The natural log is the area shown in this graph:



For example, *log 2.5* is the area is between 1 and 2.5.

Scaling Property of Logarithm

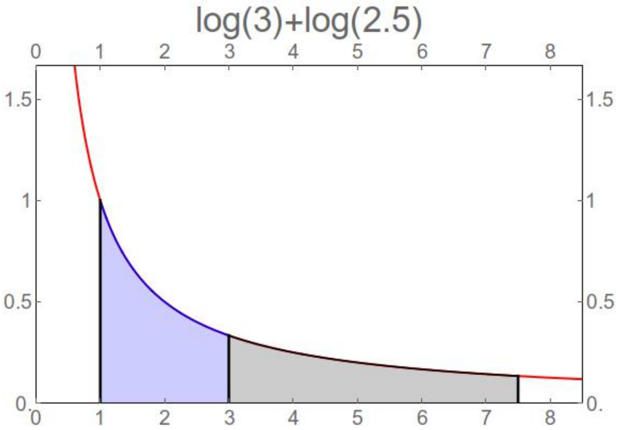
If x is scaled up by 3, then y is scaled down by 3:



Log 2.5 is also the area is between 3 and 7.5.

The area between 1 and 7.5

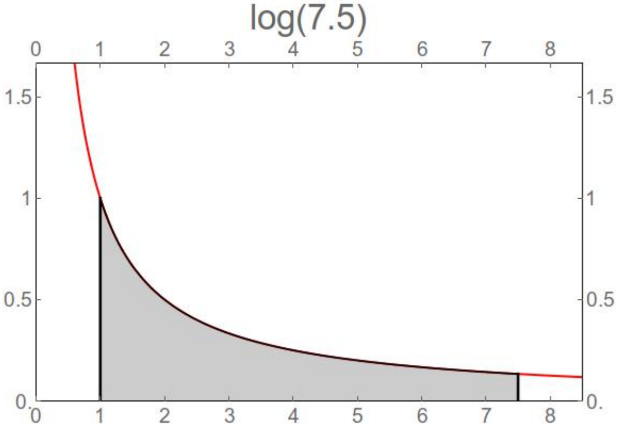
The area between 1 and 3 is just log 3.



The total area is log 3 + log 2.5

The Area between 1 and 7.5

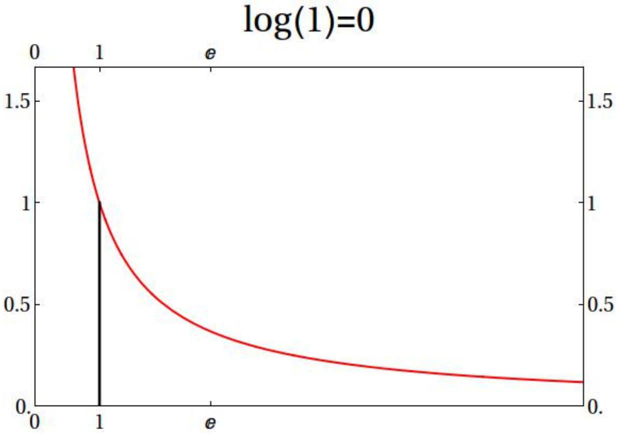
But it is also log 7.5



Therefore $\log 3 + \log 2.5 = \log 7.5$**.**

What happens if x = 1 ?

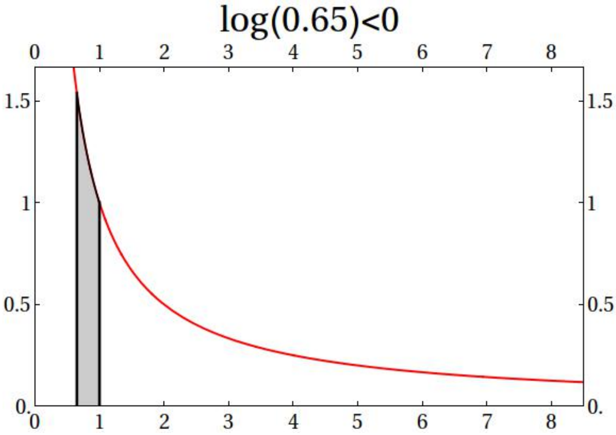
For x = 1, the area between x and 1 is zero:



Therefore log 1 = 0.

What about log x if x < 1 ?

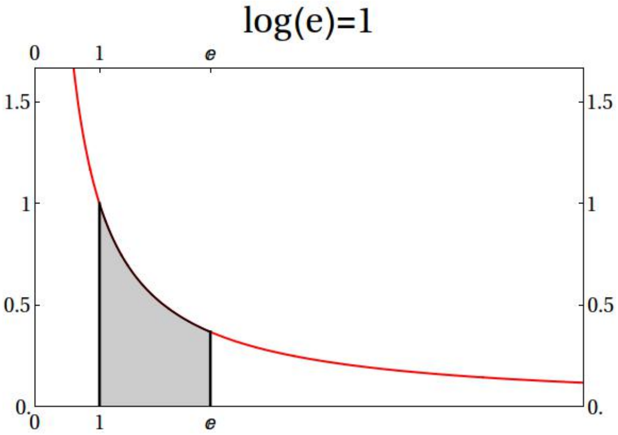
For x < 1, we need the area between x and 1.



We count this area as negative. So log x < 0.

What Number has Natural Logarithm 1?

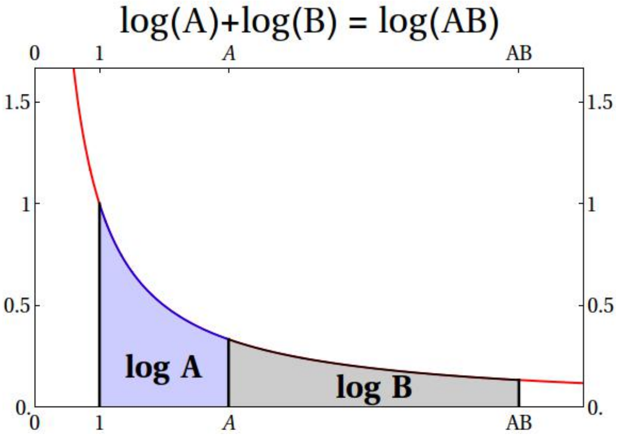
There is a number that makes the area equal to one:



This is Euler's number e. ***We will return to it later.***

Crucial Property of logs

We found the important property of logarithms:



It turns multiplication into addition.

Crucial Property of logs

We found that

$$\log 3 + \log 2.5 = \log 7.5$$

More generally,

$$\log A + \log B = \log A\,B$$

This is the most important property of logarithms:

It turns multiplication into addition.

(We will return to this important property.)

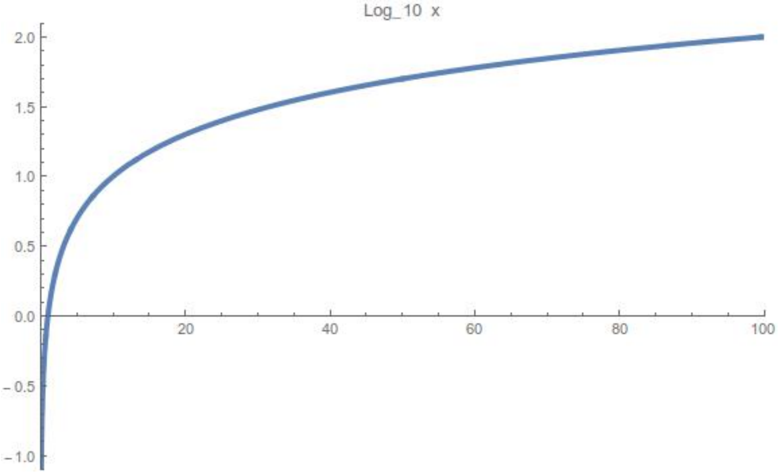
The Graph of Log x

Since $1/x$ **decreases as** x **grows, the area under the curve** $y = 1/x$ **grows very more slowly with** x**.**

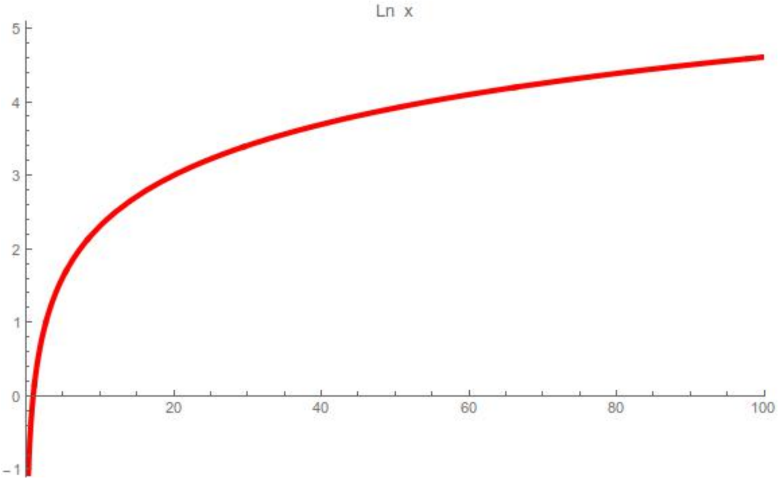
Therefore, $\log x$ **grows ever more slowly with** x**.**

Let's look at a graph of $\log x$**.**

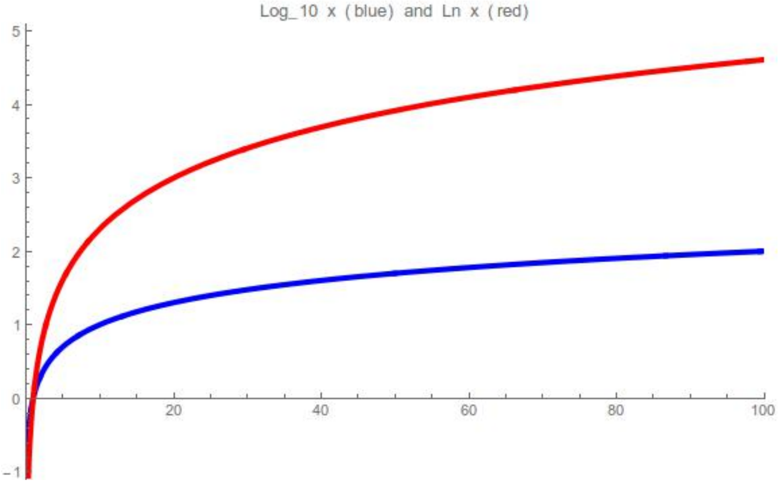
Log_{10} x for 0 < x < 100



$\mathbf{Log}_e$ x for 0 < x < 100



Log_e x and Log_10 x for 0 < x < 100



Note that $\log_e x$ is a multiple of $\log_{10} x$.

A Little Puzzle

Which is bigger:

A Googol   or   100!

Remember:

$$1 \text{ googol} = 10^{100} \qquad 100! = 1 \times 2 \times 3 \times \cdots \times 100$$

Answer to follow, but try it yourself.

Thank you