AweSums:

The Majesty of Mathematics

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Evening Course, UCD, Autumn 2016

Outline

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AweSums: The Majesty of Maths

Bernhard Riemann (1826-66)

AweSums: The Majesty of Maths

We aim to get a flavour of the Riemann Hypothesis.

It involves the zeros of the “Zeta function”:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Leonhard Euler proved the amazing result:

$$\sum_{n\in\mathbb{N}} \frac{1}{n^s} = \prod_{p\in\mathbb{P}} \left(\frac{1}{1-p^{-s}} \right)$$

This connects $\zeta(s)$ with the prime numbers.

AweSums: The Majesty of Maths

Euler's result:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(\frac{1}{1 - p^{-s}} \right)$$

The left side is the Riemann zeta function.
It is an infinite sum.

The right side is an infinite product.
There is a factor *for each prime number.*

AweSums: The Majesty of Maths

The Riemann Hypothesis is intimately connected with the distribution of the prime numbers.

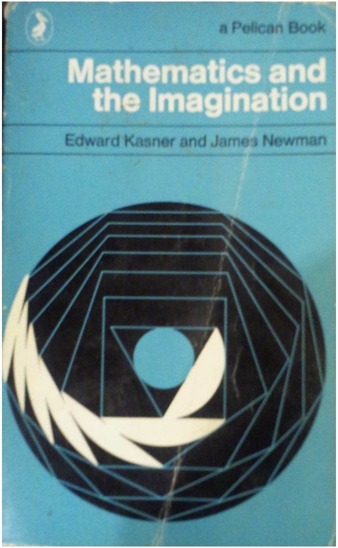
We must investigate the prime numbers and study some of their properties.

We do this in tonight's lecture.

But first we study powers of numbers.

Outline

A Googol



A nine-yer-old nephew of Edward Kasner coined the name googol for the number 1 followed by 100 zeros.

Kasner and Newman made the name popular in this book, published in 1940.

The name Google comes from a number-name coined by a 9 year old kid in 1938.

He called it a googol, and meant a 1 followed by a hundred zeros:

10 000 000 000 000 000 000 000 000 000 000 000
000 000 000 000 000 000 000 000 000 000 000
000 000 000 000 000 000 000 000 000 000 000

This is long-winded to write, and it is cumbersome to count all those zeros.

There is a better way: *Exponential Notation.*

Distraction: Wannabe Millionaire

2001: Major Charles Ingram cheats on the programme Who Wants to be a Millionaire?

Question for One Million pounds:

The number one followed by 100 zeros is called

$$\begin{array}{ll} (a)\ \textit{Googol} & (b)\ \textit{Megatron} \\ (c)\ \textit{Gigabit} & (d)\ \textit{Nanomole} \end{array}$$

After a very long time, and much coughing from an audience member, Major Ingram answered "a".

Correct. But he never collected the money!

Multiplying Powers

We can denote large numbers by using *exponents:*

3×3	**is written**	3^2
$5 \times 5 \times 5$	**is written**	5^3
$7 \times 7 \times 7 \times 7$	**is written**	7^4

What is $7^2 \times 7^3$ equal to?

$$7^2 \times 7^3 = (7 \times 7) \times (7 \times 7 \times 7) = 7 \times 7 \times 7 \times 7 \times 7 = 7^5$$

Thus,

$$7^2 \times 7^3 = 7^{2+3}$$

Note: Multiply on left. Add on right.

***More generally*, we can write**

$$7^m \times 7^n = 7^{m+n}$$

where m and n are any whole numbers.

Still *more generally*, we can write

$$x^m \times x^n = x^{m+n}$$

where x is any number.

Even *more generally*, we can write

$$x^a \times x^b = x^{a+b}$$

where x, a and b are any numbers.

Dividing Powers

Now let us look at dividing powers.

$$7^5 \div 7^2 = \frac{7^5}{7^2} = \frac{7 \times 7 \times 7 \times 7 \times 7}{7 \times 7} = 7^3$$

So we see that

$$7^5 \div 7^2 = 7^{5-2}$$

More generally

$$7^m \div 7^n = 7^{m-n}$$

Still *more generally*

$$x^m \div x^n = x^{m-n}$$

Zero-th Power of a Number

Any number divided by itself is equal to 1.

For example,

$$8^5 \div 8^5 = \frac{8^5}{8^5} = 8^{5-5} = 8^0$$

But the value of this is 1. Thus, $8^0 = 1$**.**

This is true for any number, so we have

$$x^0 = 1$$

The case 0^0 **is an exception. We do not define this.**

Negative Powers

We have already seen that

$$x^m \div x^n = x^{m-n}$$

Now suppose $m = 0$**. We get a purely negative power:**

$$x^{-n} = \frac{1}{x^n}$$

In particular,

$$x^{-1} = \frac{1}{x}$$

Fractional or Rational Powers

What are we to make of $x^{m/n}$?

Let us apply our rule to $x^{1/2}$:

$$x^{1/2} \times x^{1/2} = x^{(1/2+1/2)} = x^1 = x$$

So it is completely logical and consistent to have

$$x^{\frac{1}{2}} = \sqrt{x}$$

More generally, for $q \in \mathbb{Q}$ with $q = m/n$ we have

$$x^q = x^{m/n} = (\sqrt[n]{x})^m = (\sqrt[n]{x^m}) \,.$$

Rational & Real Powers

Now we can make sense of rational powers:

$$x^q \quad \textbf{for} \quad q \in \mathbb{Q}$$

What about irrational powers?

We can define these by a *limit process.*
Then we have a definition for all real powers

$$x^r \quad \textbf{for} \quad r \in \mathbb{R}$$

Back to a Googol (or Google)

We defined a googol to be 1 followed by 100 zeros.

Now we can write this huge number simply as

$$\text{1 googol} = 10^{100}$$

which Is remarkably compact.

$$\begin{aligned}
\text{One hundred} &= 10^2 \\
\text{One thousand} &= 10^3 \\
\text{One million} &= 10^6 \\
\text{One billion} &= 10^9 \\
\text{One trillion} &= 10^{12}
\end{aligned}$$

A Googolplex

Kasner and Newman also defined a googolplex:
It is 1 followed by a googol zeros.

We could never write it without superscript notation.

$$\textbf{1 googolplex} = 10^{\textbf{googol}} = 10^{(10^{100})}$$

This is a vast quantity, far beyond any of the constants appearing in the physical sciences.

The corporate headquarters of Google Inc. in California is called the Googleplex.

Outline

The Greek Alphabet, Part 5

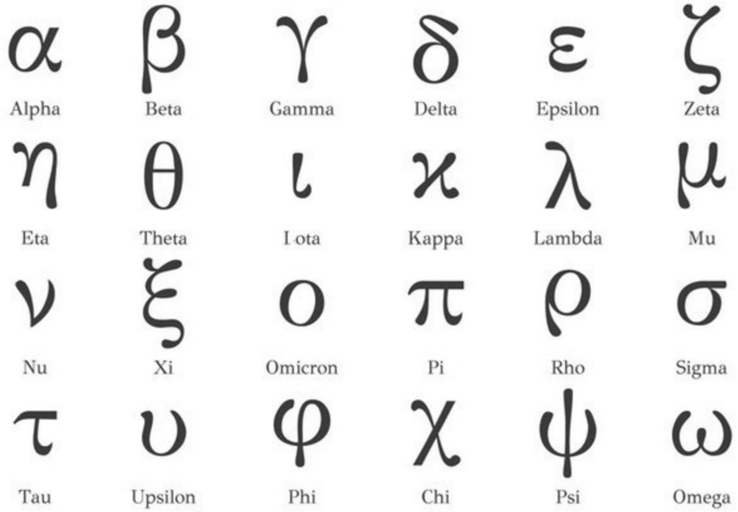


Figure : 24 beautiful letters

The Full Alphabet

α	β	γ	δ	ϵ	ζ
A	B	Γ	Δ	E	Z
η	θ	ι	κ	λ	μ
H	Θ	I	K	Λ	M
ν	ξ	o	π	ρ	σ
N	Ξ	O	Π	P	Σ
τ	υ	ϕ	χ	ψ	ω
T	Υ	Φ	X	Ψ	Ω

The Full Monty

Letter	Name	Sound	
		Ancient[5]	Modern[6]
Α α	alpha, άλφα	[a] [aː]	[a]
Β β	beta, βήτα	[b]	[v]
Γ γ	gamma, γάμμα	[g], [ŋ][7]	[ɣ] ~ [ʝ], [ŋ][8] ~ [ɲ][9]
Δ δ	delta, δέλτα	[d]	[ð]
Ε ε	epsilon, έψιλον	[e]	[e]
Ζ ζ	zeta, ζήτα	[zd][A]	[z]
Η η	eta, ήτα	[ɛː]	[i]
Θ θ	theta, θήτα	[tʰ]	[θ]
Ι ι	iota, ιώτα	[i] [iː]	[i], [ʝ],[10] [ɲ][11]
Κ κ	kappa, κάππα	[k]	[k] ~ [c]
Λ λ	lambda, λάμδα	[l]	[l]
Μ μ	mu, μυ	[m]	[m]

Letter	Name	Sound	
		Ancient[5]	Modern[6]
Ν ν	nu, νυ	[n]	[n]
Ξ ξ	xi, ξι	[ks]	[ks]
Ο ο	omicron, όμικρον	[o]	[o]
Π π	pi, πι	[p]	[p]
Ρ ρ	rho, ρώ	[r]	[r]
Σ σ/ς[13]	sigma, σίγμα	[s]	[s]
Τ τ	tau, ταυ	[t]	[t]
Υ υ	upsilon, ύψιλον	[y] [yː]	[i]
Φ φ	phi, φι	[pʰ]	[f]
Χ χ	chi, χι	[kʰ]	[x] ~ [ç]
Ψ ψ	psi, ψι	[ps]	[ps]
Ω ω	omega, ωμέγα	[ɔː]	[o]

Figure : Wikipedia: “Greek Alphabet”

A Few Greek Words With Large Letters

’ΕΛΛΑΣ
ΠΛΑΤΟΝ
ΑΚΡΟΠΟΛΙΣ

ΑΡΙΣΤΟΤΕΛΗΣ
ΠΥΘΑΓΌΡΑΣ
ΣΟΦΟΚΛΗΣ

HELLAS: **ÉΛΛΑΣ**
PLATO: **ΠΛΑΤΟΝ**
ACROPOLIS: **ΑΚΡΟΠΟΛΙΣ**

ARISTOTLE: **ΑΡΙΣΤΟΤΕΛΗΣ**
PYTHAGORAS: **ΠΥΘΑΓΌΡΑΣ**
SOPHOCLES: **ΣΟΦΟΚΛΗΣ**

Robinson’s Anemometer on East Pier

Figure : Inscription on Church in Sean McDermott Street

I asked *Cosetta Cadau,* Department of Classics Trinity College Dublin about this inscription.

Here is how she replied:

The text is not complete (the last word is cut), but what I can read is

$$MON\Omega \quad \Sigma O\Phi\Omega \quad \Theta E\Omega$$

$$\Sigma\Omega THPI \quad HM\Omega N$$

which can be translated as
To God, Our Only Saviour

End of Greek 105

Collect Your Diploma

Your Diploma

Awe Sums

ΔΙΠΛΩΜΑ

Αυτό το δίπλωμα απονέμεται στον/στην:

....

που έχει μάθει το ελληνικό αλφάβητο και μπορεί να μεταγράφει ονόματα ανθρώπων και τόπων από το ελληνικό προς το λατινικό αλφάβητο. Συγχαρητήρια.

This diploma is awarded to
(=== NAME ===)
who has learned the Greek alphabet and who can transliterate names of people and places from the Greek to the Roman alphabet.

Congratulations.

Outline

A First Look at Logs

We have

$$10^1 = 1 \qquad \textbf{and} \qquad 10^2 = 100$$

Can we find a power for numbers between 10 and 100? Can we find a number x with

$$10^x = 30$$

We can expect

- **That x is between 1 and 2**
- **Or that $1 < x < 2$**
- **Or that $x \in (1, 2)$.**

We seek x such that $10^x = 30$. Let us try $x = 1.5$:

$$10^{1.5} = 10^{1+0.5} = 10^1 \times 10^{\frac{1}{2}} = 10\sqrt{10} \approx 31.6$$

By trial-and-error or by using a calculator (or by using *WolframAlpha*) we find that

$$10^{1.477} \approx 30$$

The number 1.477 is called the logarithm of 30.

DEFINITION:
The (base 10) logarithm of a number is the power of 10 that is equal to the number.

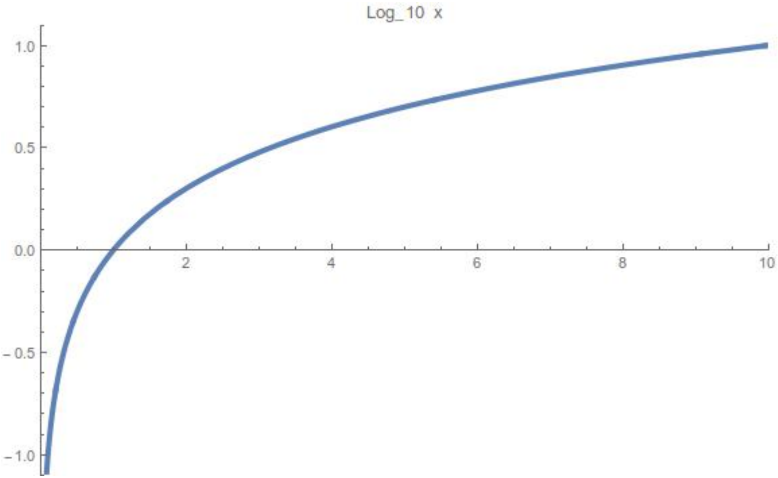
Definition of Log_{10} x

DEFINITION: **The logarithm of x is the power to which 10 must be raised to give x:**

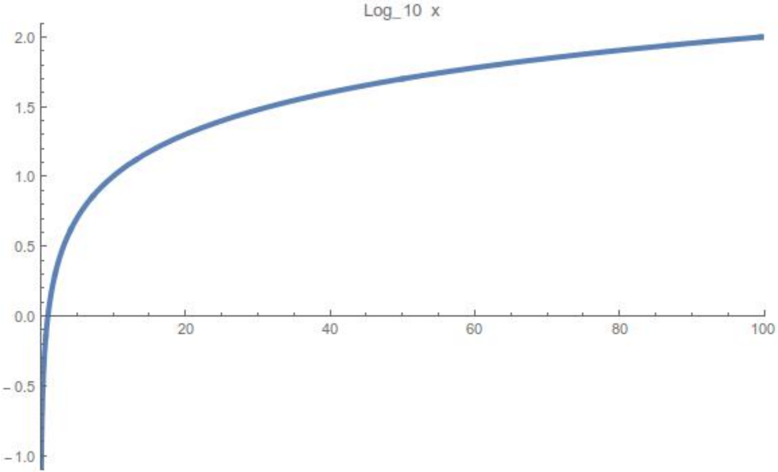
$$\log_{10} y = x \quad \Longleftrightarrow \quad 10^x = y$$

$100 = 10^2$	**so that**	$\log_{10} 100 = 2$
$1\,000 = 10^3$	**so that**	$\log_{10} 1\,000 = 3$
$1\,000\,000 = 10^6$	**so that**	$\log_{10} 1\,000\,000 = 6$
$1 = 10^0$	**so that**	$\log_{10} 1 = 0$
$\sqrt{10} = 10^{\frac{1}{2}}$	**so that**	$\log_{10} \sqrt{10} = \frac{1}{2}$
$30 = 10^{1.477}$	**so that**	$\log_{10} 30 = 1.477$

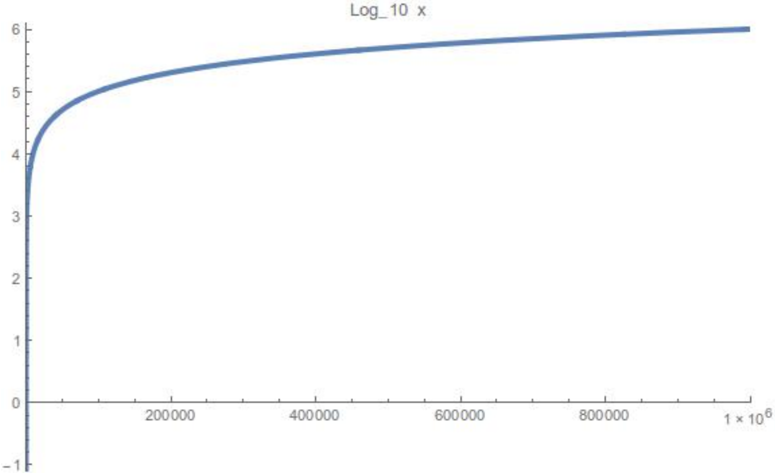
$\text{Log}_{10}\ x$ for $0 < x < 10$



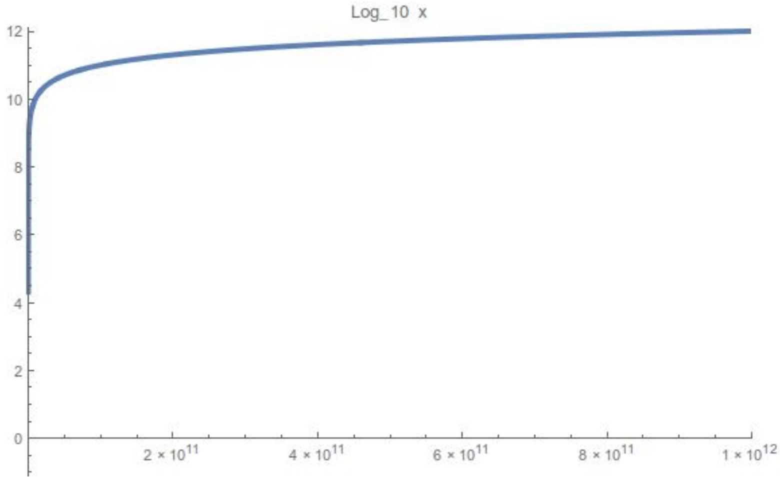
$\text{Log}_{10}\ x$ for $0 < x < 100$



$\text{Log}_{10}\, x$ for $0 < x < 10^6$



$\text{Log}_{10}\, x$ for $0 < x < 10^{12}$



Outline

Prime & Composite Numbers

A prime number is a number that cannot be broken into a product of two smaller numbers.

The first few primes are 2, 3, 5, 7, 11, 13, 17 and 19.

There are 25 primes less than 100.

Numbers that are not prime are called composite. They can be expressed as *products of primes.*

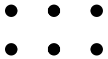
Thus, $6 = 2 \times 3$ **is a composite number.**

The number 1 **is neither prime nor composite.**

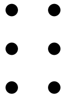
The Atoms of the Number System

A line of six spots

can be arranged in a rectangular array:



or



Note that

$$2 \times 3 = 3 \times 2$$

This is the *commutative law of multiplication.*

The Atoms of the Number System

The primes play a role in mathematics analogous to the elements of Mendeleev's Periodic Table.

Just as a chemical molecule can be constructed from the 100 or so fundamental elements, any whole number be constructed by combining prime numbers.

The primes 2, 3, 5 are the hydrogen, helium and lithium of the number system.

Some History

In 1792 Carl Friedrich Gauss, then only 15 years old, found that the proportion of primes less that n decreased approximately as $1/\log n$.

Around 1795 Adrien-Marie Legendre noticed a similar logarithmic pattern of the primes, but it was to take another century before a proof emerged.

In a letter written in 1823 the Norwegian mathematician Niels Henrik Abel described the distribution of primes as *the most remarkable result in all of mathematics.*

Percentage of Primes Less than N

Table : Percentage of Primes less than N

100	25	25.0%
1,000	168	16.8%
1,000,000	78,498	7.8%
1,000,000,000	50,847,534	5.1%
1,000,000,000,000	37,607,912,018	3.8%

We can see that the percentage of primes is falling off with increasing size.

But the rate of decrease is very slow.

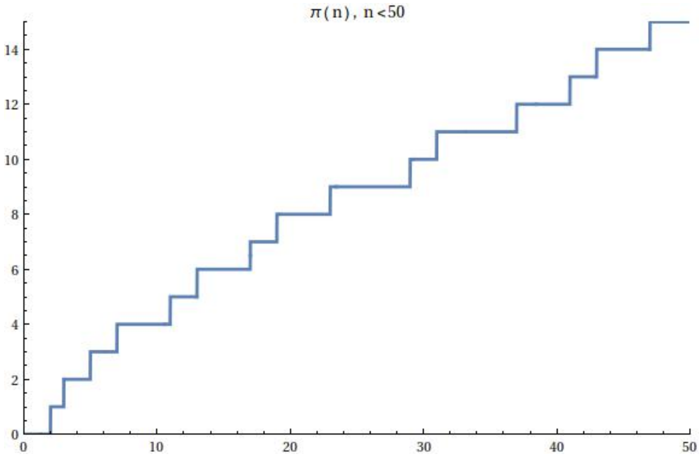


Figure : The prime counting function $\pi(n)$ for $0 \leq n \leq 50$.

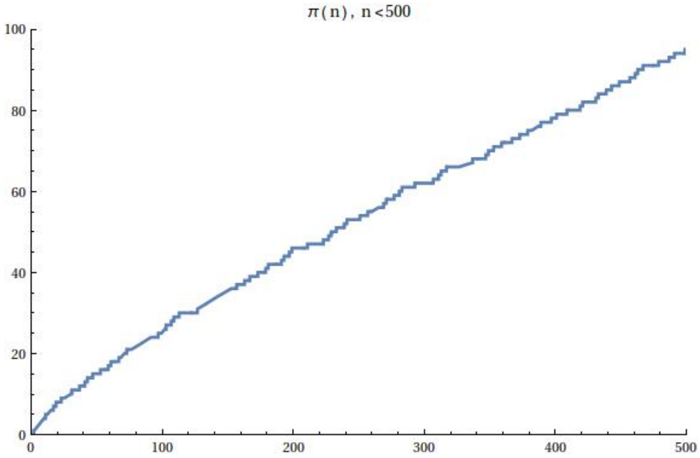


Figure : The prime counting function $\pi(n)$ for $0 \leq n \leq 500$.

Is There a Pattern in the Primes?

It is a simple matter to make a list of all the primes less that 100 or 1000.

It becomes clear very soon that there is no clear pattern emerging.

The primes appear to be scattered at random.

Figure : Prime numbers up to 100

Is There a Pattern in the Primes?

Do the primes settle down as *n* becomes larger?

Between 9,999,900 and 10,000,000
(100 numbers) there are 9 primes.

Between 10,000,000 and 10,000,100
(100 numbers) there are just 2 primes.

What kind of function could generate this behaviour?

We just do not know.

Is There a Pattern in the Primes?

The gaps between primes are very irregular.

- Can we find a pattern in the primes?
- Can we find a formula that generates primes?
- How can we determine the hundreth prime?
- What is the thousanth? The millionth?

WolframAlpha©

WolframAlpha is a Computational Knowledge Engine.

Wolfram Alpha is based on Wolfram's flagship product Mathematica, a computational platform or toolkit that encompasses computer algebra, symbolic and numerical computation, visualization, and statistics.

It is freely available through a web browser.

Euler's Formula for Primes

No mathematician has ever found a *useful* formula that generates all the prime numbers.

Euler found a beautiful little formula:

$$n^2 - n + 41$$

This gives prime numbers for n between 1 and 40.

But for $n = 41$ we get

$$41^2 - 41 + 41 = 41 \times 41$$

a composite number.

The Infinitude of Primes

Euclid proved that there is no finite limit to the number of primes.

His proof is a masterpiece of symplicity.

(See Dunham book).

Some Unsolved Problems

There appear to be an infinite number of prime pairs

$$(2n - 1, 2n + 1)$$

There are also gaps of arbitrary length:

for example, there are 13 consecutive composite numbers between 114 and 126.

We can find gaps as large as we like:

Show that *N! + 1* is followed by a sequence of *N - 1* composite numbers.

Primes have been used as markers of civilization.

In the novel Cosmos, by Carl Sagan,
the heroine detects a signal:

- **First 2 pulses**
- **Then 3 pulses**
- **Then 5 pulses**
- **. . .**
- **Then 907 pulses.**

In each case, a prime number of pulses.
This could hardly be due to any natural phenomenon.

Is this evidence of extra-terrestrial intelligence?

Thank you