AweSums:

The Majesty of Mathematics

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Evening Course, UCD, Autumn 2016

Outline

Introduction

Hilbert's Problems

Irrational Numbers

Greek 4

The Real Number Line

Carl Friedrich Gauss

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AweSums: The Majesty of Maths

Bernhard Riemann (1826-66)

AweSums: The Majesty of Maths

We aim to get a flavour of the Riemann Hypothesis.

It involves the zeros of the "Zeta function":

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

AweSums: The Majesty of Maths

We aim to get a flavour of the Riemann Hypothesis.

It involves the zeros of the "Zeta function":

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

The symbol s represents a complex number.

To get to the complex numbers, we are building up the hierarchy of numbers:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

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Hilbert's Problems

In August 1900, David Hilbert addresed the International Congress of Mathematicians in the Sorbonne in Paris:

> ***"Who of us would not be glad to lift the veil behind which the future lies hidden; to cast a glance at the next advances of our science and at the secrets of its development during future centuries?"***

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Hilbert presented 23 problems that challenged mathematicians through the twentieth century.

Hilbert's Problems

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 37, Number 4, Pages 407–436
S 0273-0979(00)00881-8
Article electronically published on June 26, 2000

MATHEMATICAL PROBLEMS

DAVID HILBERT

Lecture delivered before the International Congress of Mathematicians at Paris in 1900.

Hilbert's eighth problem concerned itself with what is called the Riemann Hypothesis (RH).

RH is generally regarded as the deepest and most important unproven mathematical problem.

Anyone who can prove it is assured of lasting fame.

Why is RH Important?

A large number of mathematical theorems (1000's) depend for their validity on the RH.

Were RH to turn out to be false, many of these mathematical arguments would simply collapse.

In 2000, industrialist Landon Clay donated $7M, with $1M for each of 7 problems in mathematics.

The Riemann hypothesis is one of these problems.

Why is RH Important?

Whoever proves Riemann's hypothesis will have completed thousands of theorems that start like this:

"Assuming that the Riemann hypothesis is true . . . ".

He or she will be assured of lasting fame.

Those who establish fundamental mathematical results probably come closer to immortality than almost anyone else.

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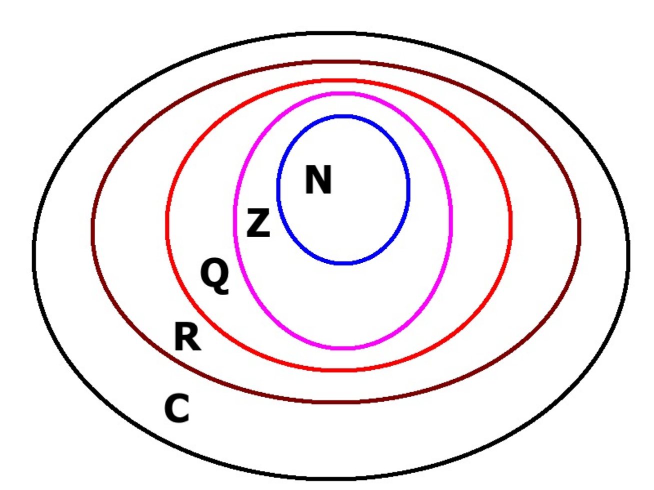
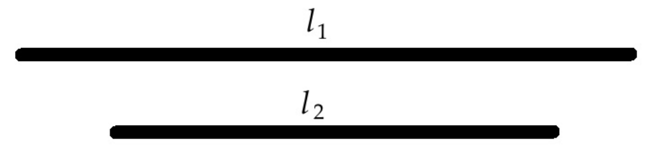
The Hierarchy of Numbers



$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

Incommensurability

Suppose we have two line segments



Can we find a unit of measurement such that both lines are a whole number of units?

Can they be co-measured? Are they commensurable?

Are ℓ_1 and ℓ_2 commensurable?

If so, let the unit of measurement be λ.

Then

$$\begin{aligned} \ell_1 &= m\lambda\,, \quad m \in \mathbb{N} \\ \ell_2 &= n\lambda\,, \quad n \in \mathbb{N} \end{aligned}$$

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$$\frac{\ell_1}{\ell_2} = \frac{m\lambda}{n\lambda} = \frac{m}{n}$$

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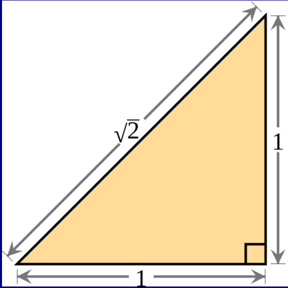
Therefore

$$\frac{\ell_1}{\ell_2} = \frac{m\lambda}{n\lambda} = \frac{m}{n}$$

If not, then ℓ_1 and ℓ_2 are incommensurable.

Irrational Numbers

If the side of a square is of length 1, then the diagonal has length $\sqrt{2}$ (by the Theorem of Pythagoras).



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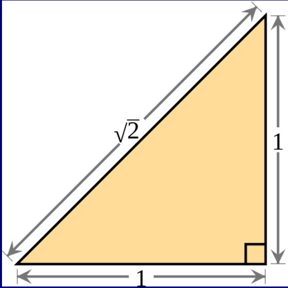

The ratio between the diagonal and the side is:

$$\frac{\text{Diagonal}}{\text{Side Length}} = \sqrt{2}$$

Irrationality of $\sqrt{2}$

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1. Whole numbers
2. Ratios of whole numbers

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For example, suppose $p = 42$ and $q = 30$. Then

$$\frac{p}{q} = \frac{42}{30} = \frac{7 \times 6}{5 \times 6} = \frac{7}{5}$$

We say that p and q are relatively prime:
They have no common factors.

In particular, p and q cannot both be even numbers.

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Now square both sides of the equation $\sqrt{2} = p/q$:

$$2 = \frac{p}{q} \times \frac{p}{q} = \frac{p^2}{q^2} \qquad \text{or} \qquad p^2 = 2q^2$$

This means that p^2 is even. Therefore p is even.

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But this means that q^2 is even. So, q is even.

Both p and q are even. This is a contradiction.

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$\kappa\rho\iota\sigma\eta$ $\kappa\alpha\tau\alpha\sigma\tau\rho o\phi\eta$!

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- **Greek 4**
- The Real Number Line
- Carl Friedrich Gauss

The Greek Alphabet, Part 4

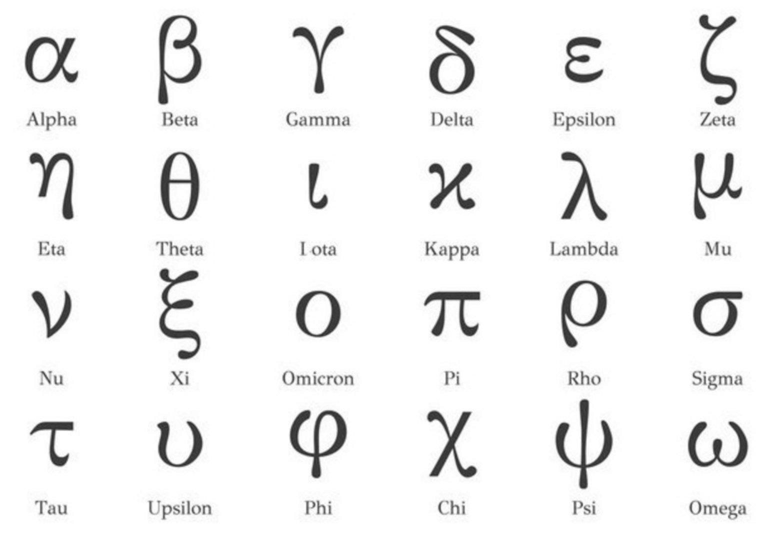

Figure : 24 beautiful letters

The Last Six Letters

We will consider the final group of six letters.

$$\tau \quad \upsilon \quad \phi \quad \chi \quad \psi \quad \omega$$

$$\mathrm{T} \quad \Upsilon \quad \Phi \quad \mathrm{X} \quad \Psi \quad \Omega$$

Let us focus first on the small letters
and come back to the big ones later.

$\tau \quad \upsilon \quad \phi \quad \chi \quad \psi \quad \omega$

Tau: You have certainly heard of a Tau-cross: τ.

Upsilon (υ) or u-psilon means ‘bare u’.
It is often transliterated as ‘y’.

Phi (ϕ) is ‘ f ’, often used for latitude
(as λ is often used for longitude).

Chi (χ) has a ‘ch’ or ‘k’ sound.

Psi (ψ) is very common: psychology, etc.

Omega (ω) is the end: Alpha and Omega $\left(\frac{\mathrm{A}}{\Omega}\right)$.

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Now you know 24 letters. You should get a diploma.

A Few Greek Words (for practice)

$\kappa\omega\mu\alpha$
$\psi\upsilon\kappa\eta$
$\kappa\rho\iota\sigma\iota\varsigma$

$\alpha\nu\alpha\theta\epsilon\mu\alpha$
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Coma: $\kappa\omega\mu\alpha$
Psyche: $\psi\upsilon\kappa\eta$
Crisis: $\kappa\rho\iota\sigma\iota\varsigma$

Anathema: $\alpha\nu\alpha\theta\epsilon\mu\alpha$
Ambrosia: $\alpha\mu\beta\rho o\sigma\iota\alpha$
Catastrophe: $\kappa\alpha\tau\alpha\sigma\tau\rho o\phi\eta$


Αεροδρόμιο
Airport
Ξενοφώντος
Ομόνοια
Μητροπόλεως
Ερμού
Πειραιώς
Β. Σοφίας


Ο Θεός
Εκκλησία


Ο Θεός
God
Εκκλησία
Church

End of Greek 104

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The Real Numbers

WARNING: THIS SECTION IS TOUGH !!!

The Real Numbers

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It is as if there are gaps in the number system.

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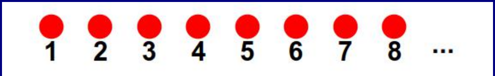
The Pythagoreans found that no number known to them gave the diagonal of a unit square.

It is as if there are gaps in the number system.

We look at the rational numbers and show how to complete them: how to fill in the gaps.

The set $\mathbb{N}$ is infinite, but each element is isolated.



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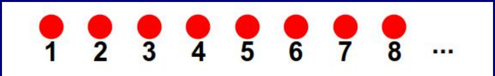

The set $\mathbb{Q}$ is infinite and also dense:
between any two rationals there is another rational.

PROOF: Let $r_1 = p_1/q_1$ and $r_2 = p_2/q_2$ be rationals.

$$\bar{r} = \tfrac{1}{2}(r_1 + r_2) = \frac{1}{2}\left(\frac{p_1}{q_1} + \frac{p_2}{q_2}\right) = \frac{p_1 q_2 + q_1 p_2}{2 q_1 q_2}$$

is another rational between them: $r_1 < \bar{r} < r_2$.

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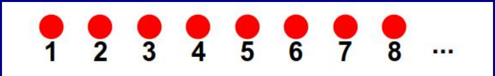

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We complete it—filling in the gaps—by defining the limit of any sequence of rationals as a real number.

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WARNING:
We are glossing over a number of fundamental ideas of mathematical analysis:

- **What is an infinite sequence?**
- **What is the limit of a sequence?**

We will return later to these ideas.

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$$\sqrt{2} = 1.41421356\ldots$$

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In terms of fractions, this is the sequence

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EXERCISE:
Construct a sequence in $\mathbb{Q}$ that tends to π.

The Real Number Line

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PHYSICS: There are unknown aspects of the microscopic structure of spacetime! These go beyond our 'Universe of Discourse'.

Now we have the chain of sets:

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We will soon talk about prime numbers $\mathbb{P}$.
They are subset of the natural numbers, so

$$\mathbb{P} \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{A} \subset \mathbb{R}$$

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Carl Friedrich Gauss

Carl Friedrich Gauss (1777–1855)

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A German mathematician who made profound contributions to many fields of mathematics:

- Number theory
- Algebra
- Statistics
- Analysis
- Differential geometry
- Geodesy & Geophysics
- Mechanics & Electrostatics
- Astronomy

Gauss is regarded as one of the greatest mathematicians of all time.

Gauss Outsmarts his Teacher

Gauss was the doctoral supervisor of Riemann

Gauss was a genius. He was known as

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When very young, Gauss outsmarted his teacher.

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The teacher got his own back. Ho! ho! ho!

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Gauss's school teacher tasked the class:

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How did Gauss do it?

First, Gauss wrote the numbers in a row:

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Then he added the two rows, column by column:

$$\begin{array}{ccccccc} 1 & 2 & 3 & \ldots & 98 & 99 & 100 \\ 100 & 99 & 98 & \ldots & 3 & 2 & 1 \\ \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ \\ 101 & 101 & 101 & \ldots & 101 & 101 & 101 \end{array}$$

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Clearly, the total for the two rows is 10,100.

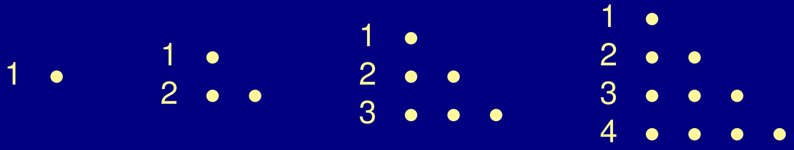
But every number from 1 to 100 is counted twice.

$$\therefore \quad 1 + 2 + 3 + \cdots + 98 + 99 + 100 = 5{,}050$$

Triangular Numbers

Gauss had calculated the 100-th triangular number.

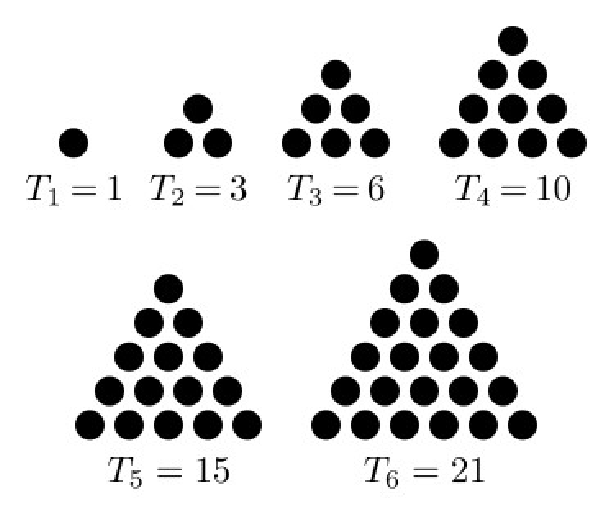
Let us take a geometrical look at the sums of the first few natural numbers:



We see that the sums can be arranged as triangles.

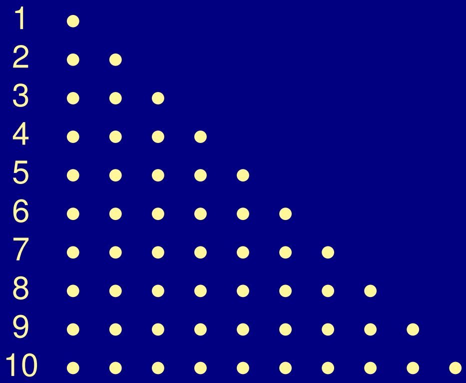
Triangular Numbers

The first few triangular numbers are $\{1, 3, 6, 10, 15, 21\}$.



Let's look at the 10th triangular number.

For $n = 10$ the pattern is:



How do we compute its value? Gauss's method!

It is easy to show that the n-th triangular number is

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We do just as Gauss did, and list the numbers twice:

$$\begin{array}{cccccc} 1 & 2 & 3 & \ldots & n-1 & n \\ n & n-1 & n-2 & \ldots & 2 & 1 \\ --- & --- & --- & \ldots & --- & --- \\ n+1 & n+1 & n+1 & \ldots & n+1 & n+1 \end{array}$$

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Each number has been counted twice, so

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Let's check this for Gauss's problem of $n = 100$:

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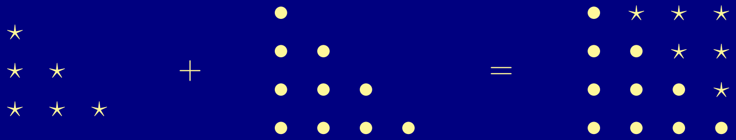
Gauss's approach was to look at the problem from a new angle.

Such *lateral thinking* is very common in mathematics:

Problems that look difficult can sometimes be solved easily when tackled from a different angle.

Two Triangles Make a Square

A nice property of *consecutive* triangular numbers:



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				●					●	⋆	⋆	⋆
⋆				●	●				●	●	⋆	⋆
⋆	⋆		+	●	●	●		=	●	●	●	⋆
⋆	⋆	⋆		●	●	●	●		●	●	●	●

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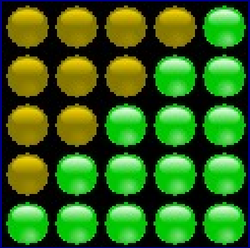

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The result is a perfect square.

Puzzle

What is the sum of all the numbers from 1 up to 100 and back down again?

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The answer is in the video coming up now.

A Video from the Museum of Mathematics

VIDEO: Beautiful Maths, available at

`http://momath.org/home/beautifulmath/`

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EXERCISE: Zink about that!

A Lateral Thinking Puzzle

- **Jill is** 23 **years younger than her father.**
- **What age was she when she was half his age?**

A Lateral Thinking Puzzle

- Jill is 23 years younger than her father.
- What age was she when she was half his age?

Hint: Be Smart
There is no need for tricky algebra.

A Lateral Thinking Puzzle

Solution later!

Thank you