AweSums:

The Majesty of Mathematics

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Outline

Introduction

Hilbert's Problems

Irrational Numbers

Greek 4

The Real Number Line

Carl Friedrich Gauss

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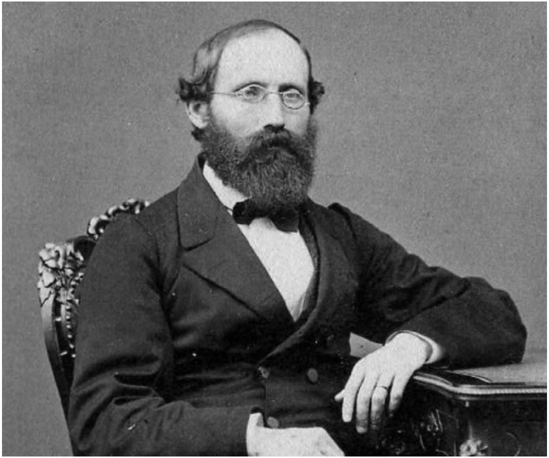
AweSums: The Majesty of Maths



Bernhard Riemann (1826-66)

AweSums: The Majesty of Maths

We aim to get a flavour of the Riemann Hypothesis.

It involves the zeros of the "Zeta function":

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

The symbol s represents a complex number.

To get to the complex numbers, we are building up the hierarchy of numbers:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

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Hilbert's Problems

In August 1900, David Hilbert addresed the *International Congress of Mathematicians* in the Sorbonne in Paris:

> ***"Who of us would not be glad to lift the veil behind which the future lies hidden; to cast a glance at the next advances of our science and at the secrets of its development during future centuries?"***

Hilbert presented 23 problems that challenged mathematicians through the twentieth century.

Hilbert's Problems

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 37, Number 4, Pages 407–436
S 0273-0979(00)00881-8
Article electronically published on June 26, 2000

MATHEMATICAL PROBLEMS

DAVID HILBERT

Lecture delivered before the International Congress of Mathematicians at Paris in 1900.

Hilbert's eighth problem concerned itself with what is called the Riemann Hypothesis (RH).

RH is generally regarded as the deepest and most important unproven mathematical problem.

Anyone who can prove it is assured of lasting fame.

Why is RH Important?

A large number of mathematical theorems (1000's) depend for their validity on the RH.

Were RH to turn out to be false, many of these mathematical arguments would simply collapse.

In 2000, industrialist Landon Clay donated $7M, with $1M for each of 7 problems in mathematics.

The Riemann hypothesis is one of these problems.

Why is RH Important?

Whoever proves Riemann's hypothesis will have completed thousands of theorems that start like this:

"Assuming that the Riemann hypothesis is true . . . ".

He or she will be assured of lasting fame.

Those who establish fundamental mathematical results probably come closer to immortality than almost anyone else.

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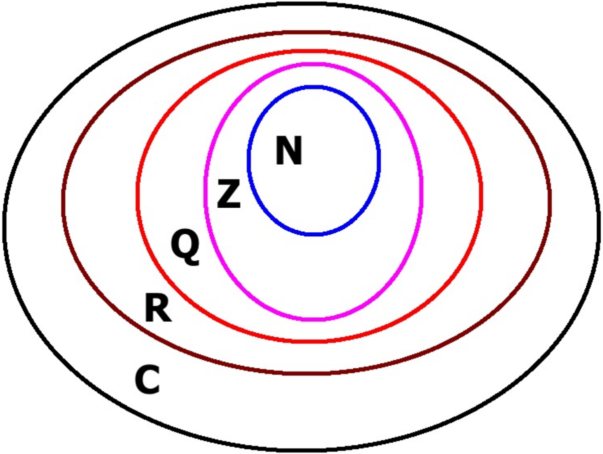
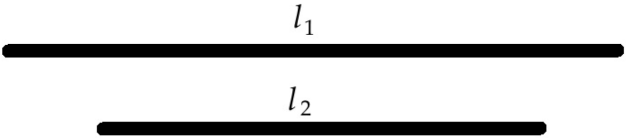
The Hierarchy of Numbers



$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

Incommensurability

Suppose we have two line segments



Can we find a unit of measurement such that
both lines are a whole number of units?

Can they be co-measured? Are they commensurable?

Are ℓ_1 and ℓ_2 commensurable?

If so, let the unit of measurement be λ.

Then

$$\begin{aligned} \ell_1 &= m\lambda\,, \quad m \in \mathbb{N} \\ \ell_2 &= n\lambda\,, \quad n \in \mathbb{N} \end{aligned}$$

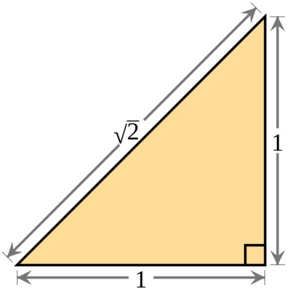
Therefore

$$\frac{\ell_1}{\ell_2} = \frac{m\lambda}{n\lambda} = \frac{m}{n}$$

If not, then ℓ_1 and ℓ_2 are incommensurable.

Irrational Numbers

If the side of a square is of length 1, then the diagonal has length $\sqrt{2}$ (by the Theorem of Pythagoras).



The ratio between the diagonal and the side is:

$$\frac{\textbf{Diagonal}}{\textbf{Side Length}} = \sqrt{2}$$

Irrationality of $\sqrt{2}$

For the Pythagoreans, numbers were of two types:

1. ***Whole numbers***
2. ***Ratios of whole numbers***

There were no other numbers.

Let's suppose that $\sqrt{2}$ is a ratio of whole numbers:

$$\sqrt{2} = \frac{p}{q}$$

We can suppose that p and q have no common factors. Otherwise, we just cancel them out.

For example, suppose $p = 42$ and $q = 30$. Then

$$\frac{p}{q} = \frac{42}{30} = \frac{7 \times 6}{5 \times 6} = \frac{7}{5}$$

We say that p and q are relatively prime:
They have no common factors.

In particular,* p *and* q *cannot both be even numbers.

Now square both sides of the equation $\sqrt{2} = p/q$:

$$2 = \frac{p}{q} \times \frac{p}{q} = \frac{p^2}{q^2} \qquad \textbf{or} \qquad p^2 = 2q^2$$

This means that p^2 is even. Therefore p is even.

Let $p = 2r$ where r is another whole number. Then

$$p^2 = (2r)^2 = 4r^2 = 2q^2 \qquad \textbf{or} \qquad 2r^2 = q^2$$

But this means that q^2 is even. So, q is even.

Both p and q are even. This is a contradiction.

The supposition was that $\sqrt{2}$ is a ratio of two integers that have no common factors.

This assumption has led to a contradiction.

By reductio ad absurdum, $\sqrt{2}$ is *irrational.*

It is not a ratio of whole numbers.

To the Pythagoreans, $\sqrt{2}$ was not a number.

$$\kappa\rho\iota\sigma\eta \qquad \kappa\alpha\tau\alpha\sigma\tau\rho o\phi\eta!$$

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The Greek Alphabet, Part 4

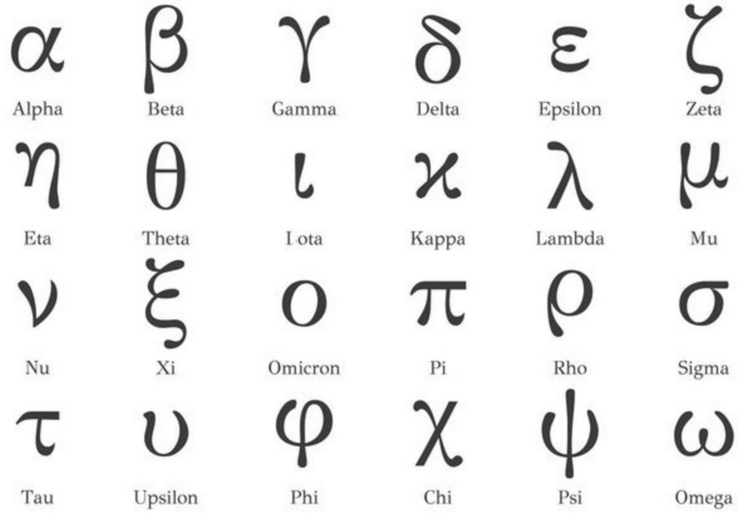

Figure : 24 beautiful letters

The Last Six Letters

We will consider the final group of six letters.

$$\tau \quad \upsilon \quad \phi \quad \chi \quad \psi \quad \omega$$

$$\mathrm{T} \quad \Upsilon \quad \Phi \quad \mathrm{X} \quad \Psi \quad \Omega$$

Let us focus first on the *small letters* and come back to the big ones later.

$$\tau \quad \upsilon \quad \phi \quad \chi \quad \psi \quad \omega$$

Tau: You have certainly heard of a Tau-cross: τ.

Upsilon (υ) or u-psilon means 'bare u'.
It is often transliterated as 'y'.

Phi (ϕ) is ' f ', often used for latitude
(as λ is often used for longitude).

Chi (χ) has a 'ch' or 'k' sound.

Psi (ψ) is very common: psychology, etc.

Omega (ω) is the end: Alpha and Omega $\left(\frac{\mathrm{A}}{\Omega}\right)$.

Now you know 24 letters. You should get a diploma.

A Few Greek Words (for practice)

$\kappa\omega\mu\alpha$
$\psi\upsilon\kappa\eta$
$\kappa\rho\iota\sigma\iota\varsigma$

$\alpha\nu\alpha\theta\epsilon\mu\alpha$
$\alpha\mu\beta\rho o\sigma\iota\alpha$
$\kappa\alpha\tau\alpha\sigma\tau\rho o\phi\eta$

Coma: $\kappa\omega\mu\alpha$
Psyche: $\psi\upsilon\kappa\eta$
Crisis: $\kappa\rho\iota\sigma\iota\varsigma$

Anathema: $\alpha\nu\alpha\theta\epsilon\mu\alpha$
Ambrosia: $\alpha\mu\beta\rho o\sigma\iota\alpha$
Catastrophe: $\kappa\alpha\tau\alpha\sigma\tau\rho o\phi\eta$


Αεροδρόμιο
Airport
Ξενοφώντος
Ομόνοια
Μητροπόλεως
Ερμού
Πειραιώς
Β. Σοφίας


Ο Θεός
Εκκλησία


Ο Θεός
God
Εκκλησία
Church

End of Greek 104

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The Real Numbers

WARNING: THIS SECTION IS TOUGH !!!

We need to be able to assign a number to a line of any length.

The Pythagoreans found that no number known to them gave the diagonal of a unit square.

It is as if there are gaps in the number system.

We look at the rational numbers and show how to complete them: how to fill in the gaps.

The set $\mathbb{N}$ is infinite, but each element is isolated.

The set $\mathbb{Q}$ is infinite and also dense:
between any two rationals there is another rational.

PROOF: Let $r_1 = p_1/q_1$ and $r_2 = p_2/q_2$ be rationals.

$$\bar{r} = \tfrac{1}{2}(r_1 + r_2) = \frac{1}{2}\left(\frac{p_1}{q_1} + \frac{p_2}{q_2}\right) = \frac{p_1 q_2 + q_1 p_2}{2 q_1 q_2}$$

is another rational between them: $r_1 < \bar{r} < r_2$.

Although $\mathbb{Q}$ is dense, there are gaps.
The line of rationals is discontinuous.

We complete it—filling in the gaps—by *defining* the limit of any sequence of rationals as a real number.

WARNING:
We are glossing over a number of fundamental ideas of mathematical analysis:

- What is an infinite sequence?
- What is the limit of a sequence?

We will return later to these ideas.

To give a particular example, we know that

$$\sqrt{2} = 1.41421356\ldots$$

We construct a sequence of rational numbers

$$\{1, 1.4, 1.41, 1.414, 1.4142, 1.41421, 1.414213, \ldots\}$$

In terms of fractions, this is the sequence

$$\left\{1, \frac{14}{10}, \frac{141}{100}, \frac{1414}{1000}, \frac{14142}{10000}, \frac{141421}{100000}, \frac{1414213}{1000000}, \ldots\right\}$$

These rational numbers get *closer and closer to* $\sqrt{2}$.

EXERCISE:
Construct a sequence in $\mathbb{Q}$ that tends to π.

The Real Number Line

The set of Real Numbers, $\mathbb{R}$, contains all the rational numbers in $\mathbb{Q}$ and also all the limits of sequences of rationals [technically, all ‘Cauchy sequences’].

We may assume that

- Every point on the number line corresponds to a real number.
- Every real number corresponds to a point on the number line.

PHYSICS: There are unknown aspects of the microscopic structure of spacetime! These go beyond our ‘Universe of Discourse’.

Now we have the chain of sets:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

The irrational numbers fall into two categories:

- **Algebraic numbers like $\sqrt{2}$.**
- **Transcendental numbers like π.**

We denote the algebraic numbers by $\mathbb{A}$.
Now we have the chain of sets:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{A} \subset \mathbb{R}$$

We will soon talk about prime numbers $\mathbb{P}$.
They are subset of the natural numbers, so

$$\mathbb{P} \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{A} \subset \mathbb{R}$$

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Carl Friedrich Gauss

Carl Friedrich Gauss (1777–1855)

Carl Friedrich Gauss (1777–1855)

A German mathematician who made profound contributions to many fields of mathematics:

- **Number theory**
- **Algebra**
- **Statistics**
- **Analysis**
- **Differential geometry**
- **Geodesy & Geophysics**
- **Mechanics & Electrostatics**
- **Astronomy**

Gauss is regarded as one of the greatest mathematicians of all time.

Gauss Outsmarts his Teacher

Gauss was the doctoral supervisor of Riemann

Gauss was a genius. He was known as

The Prince of Mathematicians.

When very young, Gauss outsmarted his teacher.

I can now reveal a fact unknown to historians:

The teacher got his own back. Ho! ho! ho!

Gauss Outsmarts his Teacher

Gauss's school teacher tasked the class:

- **Add up all the whole numbers from 1 to 100.**

Gauss solved the problem in a flash.

He wrote the correct answer,

$$5,050$$

on his slate and handed it to the teacher.

How did Gauss do it?

First, Gauss wrote the numbers in a row:

$$1 \quad 2 \quad 3 \quad \ldots \quad 98 \quad 99 \quad 100$$

Next he wrote them again, *in reverse order:*

$$\begin{array}{ccccccc} 1 & 2 & 3 & \ldots & 98 & 99 & 100 \\ 100 & 99 & 98 & \ldots & 3 & 2 & 1 \end{array}$$

Then he added the two rows, column by column:

$$\begin{array}{ccccccc} 1 & 2 & 3 & \ldots & 98 & 99 & 100 \\ 100 & 99 & 98 & \ldots & 3 & 2 & 1 \\ \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ \\ 101 & 101 & 101 & \ldots & 101 & 101 & 101 \end{array}$$

Clearly, the total for the two rows is 10,100.

But every number from 1 to 100 is counted twice.

$$\therefore\ \mathbf{1 + 2 + 3 + \cdots + 98 + 99 + 100 = 5{,}050}$$

Triangular Numbers

Gauss had calculated the 100-th triangular number.

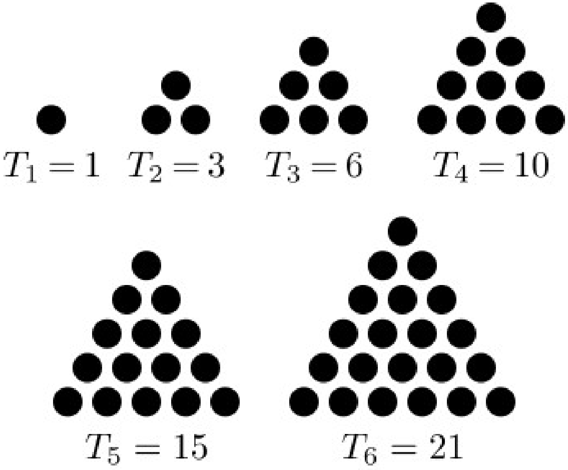
Let us take a geometrical look at the sums of the first few natural numbers:

1 •	1 •	1 •	1 •
	2 • •	2 • •	2 • •
		3 • • •	3 • • •
			4 • • • •

We see that the sums can be arranged as triangles.

Triangular Numbers

The first few *triangular numbers* are $\{1, 3, 6, 10, 15, 21\}$.



Let's look at the 10th triangular number.

For $n = 10$ the pattern is:

1	•
2	• •
3	• • •
4	• • • •
5	• • • • •
6	• • • • • •
7	• • • • • • •
8	• • • • • • • •
9	• • • • • • • • •
10	• • • • • • • • • •

How do we compute its value? Gauss's method!

It is easy to show that the n-th triangular number is

$$T_n = (1 + 2 + 3 + \cdots + n) = \frac{1}{2}n(n+1)$$

We do just as Gauss did, and list the numbers twice:

$$\begin{array}{cccccc} 1 & 2 & 3 & \ldots & n-1 & n \\ n & n-1 & n-2 & \ldots & 2 & 1 \\ --- & --- & --- & \ldots & --- & --- \\ n+1 & n+1 & n+1 & \ldots & n+1 & n+1 \end{array}$$

There are n columns, each with total $n+1$.

So the grand total is $n \times (n+1)$.

Each number has been counted twice, so

$$T_n = \frac{1}{2}n(n+1)$$

Let's check this for Gauss's problem of $n = 100$:

$$T_{100} = 1 + 2 + 3 + \cdots + 100 = \frac{100 \times 101}{2} = 5,050$$

Gauss's approach was to look at the problem from a new angle.

Such *lateral thinking* is very common in mathematics:

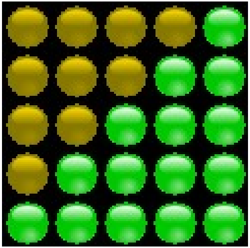
Problems that look difficult can sometimes be solved easily when tackled from a different angle.

Two Triangles Make a Square

A nice property of *consecutive* triangular numbers:

$$\begin{array}{ccc} \star & & \\ \star & \star & \\ \star & \star & \star \end{array} \quad + \quad \begin{array}{cccc} \bullet & & & \\ \bullet & \bullet & & \\ \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \bullet \end{array} \quad = \quad \begin{array}{cccc} \bullet & \star & \star & \star \\ \bullet & \bullet & \star & \star \\ \bullet & \bullet & \bullet & \star \\ \bullet & \bullet & \bullet & \bullet \end{array}$$

$$T_3 + T_4 = 6 + 10 = 16 = 4^2$$



Triangular Numbers

We have seen, by means of geometry that the sum of two consecutive triangular numbers is a square.

Now let us prove this algebraically:

$$\begin{aligned}
T_n + T_{n+1} &= \tfrac{1}{2}n(n+1) + \tfrac{1}{2}(n+1)(n+2) \\
&= \tfrac{1}{2}(n+1)[n+(n+2)] \\
&= \tfrac{1}{2}(n+1)[2(n+1)] \\
&= (n+1)^2
\end{aligned}$$

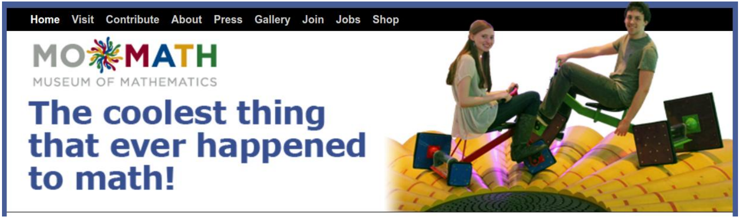
The result is *a perfect square.*

Puzzle

What is the sum of all the numbers from 1 up to 100 and back down again?

The answer is in the video coming up now.

A Video from the Museum of Mathematics



VIDEO: Beautiful Maths, available at

`http://momath.org/home/beautifulmath/`

Gauss Outsmarted by his Teacher

The teacher thought that he would have a half-hour of peace and quiet while the pupils grappled with the problem of adding up the first 100 numbers.

He was annoyed when Gauss came up almost immediately with the correct answer 5,050.

So, he said:

“Oh, you zink you are zo zmart!
Zo, multiply ze first 100 numbers.”

EXERCISE: Zink about that!

A Lateral Thinking Puzzle

- Jill is 23 years younger than her father.
- What age was she when she was half his age?

Hint: Be Smart
There is no need for tricky algebra.

A Lateral Thinking Puzzle

Solution later!

Thank you