AweSums:

The Majesty of Mathematics

Peter Lynch
School of Mathematics & Statistics
University College Dublin

Evening Course, UCD, Autumn 2016

Outline

Outline

AweSums: The Majesty of Maths

Bernhard Riemann (1826-66)

AweSums: The Majesty of Maths

We aim to get a flavour of the Riemann Hypothesis.

It involves the zeros of the "Zeta function":

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

So, we need to talk about several new topics:

- What is a function?
- What is an infinite series?
- What about convergence of a series?
- What is a complex variable?

One by one, we will look at all these questions.

But first we need to learn about the number system.

Outline

The Thallasic Age

The period from 800 BC to AD 800.

$\Theta\alpha\lambda\alpha\sigma\sigma\alpha$ — **the Sea.**

The Thallasic Age

The period from 800 BC to AD 800.

$\Theta\alpha\lambda\alpha\sigma\sigma\alpha$ — **the Sea.**

- The first Olympic Games in 776 BC
- Homer and Hesiod lived around 700 BC
- Greek mathematics began to thrive
- First two major figures: Thales and Pythagoras.

Pythagoras (c. 570–495 BC)

Pythagoras was

- Born on the island of Samos (off Turkey).
- Philosopher, mystic, prophet and religious leader.
- Contemporary with Confucius and Lao-Tzu.

Pythagoras (c. 570–495 BC)

Pythagoras was

- Born on the island of Samos (off Turkey).
- Philosopher, mystic, prophet and religious leader.
- Contemporary with Confucius and Lao-Tzu.

Words philosophy (love of learning) and mathematics (that which is learned) attributed to Pythagoras.

May have been first person to imagine that natural phenomena can be understood through mathematics.

Pythagoras (c. 570–495 BC)

- No contemporary documents
- Myth, legend and tradition
- Second or third hand accounts
  often written centuries later
- Aristotle's biography no longer extant.

Hardly any statement about Pythagoras uncontested.

Difficult to separate history from myth and legend.

Pythagoras (c. 570–495 BC)

- Travelled to Egypt, Babylon and perhaps India
- Mathematics, astronomy and religious lore
- Theorem on right-angled triangles
- Result known to Babylonians 1000 years earlier
- No record of a proof by Pythagoras survives.

The Pythagoreans

Around 530 BC Pythagoras moved to Croton in Magna Graecia (now Southern Italy).

He established an organization or school (philosophical/religious/political).

Both men and women were members of "The Pythagoreans"

Adherents were very secretive:
Bound by an oath of allegiance

Led lives of temperance; observed strict moral codes.

Pythagorean Women

"Women were given equal opportunity to study as Pythagoreans, and learned practical domestic skills in addition to philosophy.

"Women were held to be different from men, sometimes in positive ways.

"The priestess, philosopher and mathematician Themistoclea is regarded as Pythagoras' teacher; Theano, Damo and Melissa as female disciples."

From the Wikipedia article: The Pythagoreans.

Pythagorean Quotes

- I was Euphorbus at the siege of Troy.
- In anger, refrain from both speech and action.
- Educate the children and it won't be necessary to punish the men.
- Abstain from beans!

Pythagorean Quotes

- I was Euphorbus at the siege of Troy.
- In anger, refrain from both speech and action.
- Educate the children and it won't be necessary to punish the men.
- Abstain from beans!

- There is geometry in the humming of the strings,
  There is music in the spacing of the spheres.
- Number rules the universe.

Harmony & Discord

By tradition, Pythagoras discovered
the principles of musical harmony.

Stringed instruments produce harmonious sounds
when string lengths are ratios of small numbers.

Harmony & Discord

By tradition, Pythagoras discovered the principles of musical harmony.

Stringed instruments produce harmonious sounds when string lengths are ratios of small numbers.

Extended this idea to the heavens: planets emit sounds according to their speed of movement

Concept of the harmony of the spheres.

Johannes Kepler: Harmonices Mundi

“All is Number”

The motto of the Pythagoreans: “All is Number”.

All natural phenomena in the universe can be expressed using whole numbers or ratios of them.

For the Pythagoreans, numbers were the essence and source of all things.

“All is Number”

The motto of the Pythagoreans: “All is Number”.

All natural phenomena in the universe can be expressed using whole numbers or ratios of them.

For the Pythagoreans, numbers were
the essence and source of all things.

Modern physics holds that, at its deepest level, the universe is mathematical in nature.

This view is a topic of current serious discussion (The Mathematical Universe, by Max Tegmark).

Outline

Theorem of Pythagoras

The Theorem of Pythagoras is of fundamental importance in Euclidean geometry

It encapsulates the structure of space.

In the BBC series, The Ascent of Man, Jacob Bronowski said

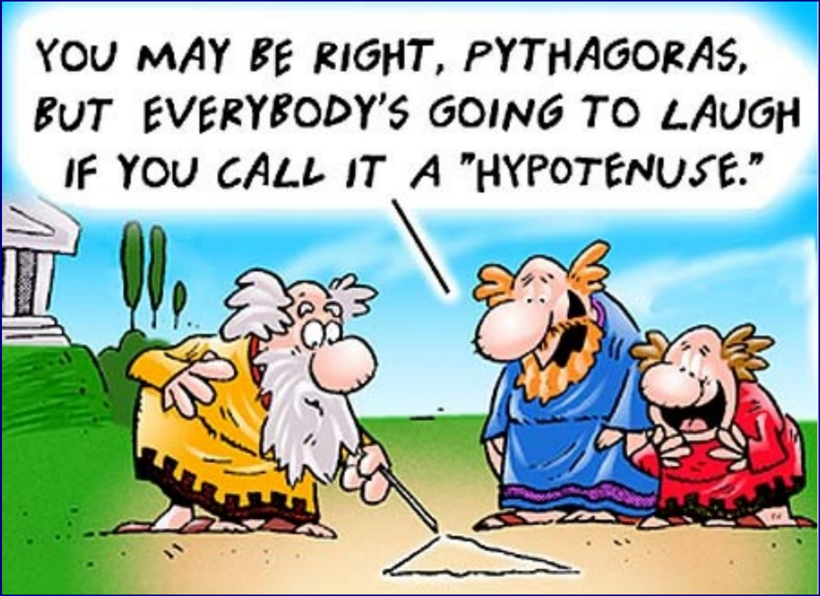
"The theorem of Pythagoras remains the most important single theorem in mathematics."

Theorem of Pythagoras

YouTube video with Danny Kaye

Google search for
"Danny Kaye Hypotenuse"

`https://www.youtube.com/watch?v=oeRCsAGQVy8`


YOU MAY BE RIGHT, PYTHAGORAS, BUT EVERYBODY'S GOING TO LAUGH IF YOU CALL IT A "HYPOTENUSE."

Hypotenuse

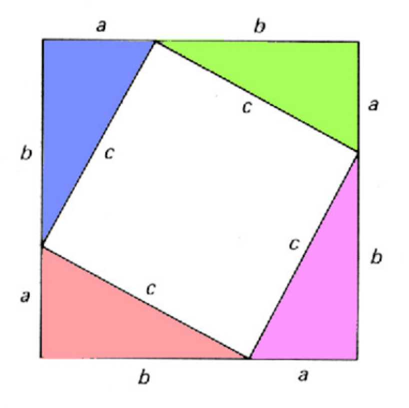
The side of a right triangle opposite to the right angle.

1570s, from Late Latin hypotenusa, from Greek hypoteinousa “stretching under" (the right angle).

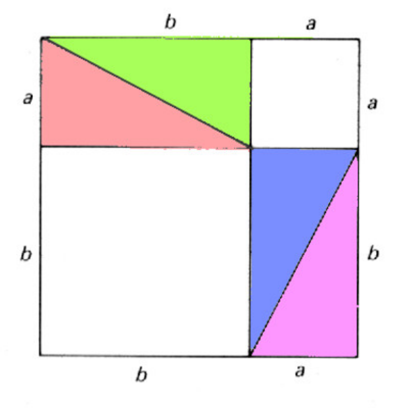
Fem. present participle of hypoteinein, from hypo- “under" + teinein “to stretch"

From Online Etymology Dictionary: `http://www.etymonline.com/`

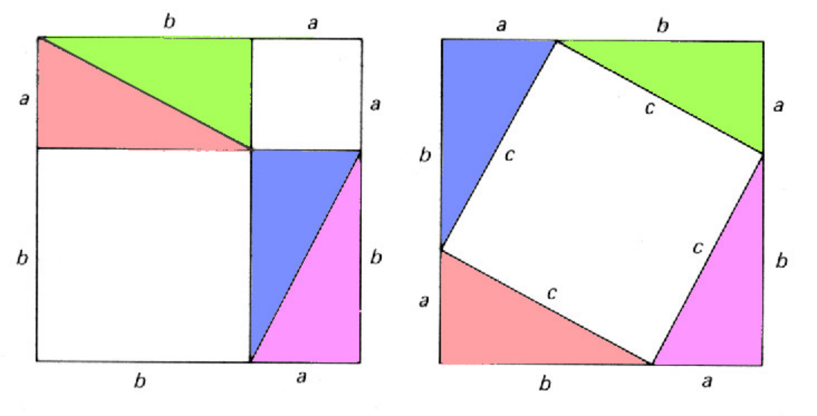
Proof without Formulae



Proof without Formulae



Proof without Formulae



$$a^2 + b^2 = c^2$$

Why is this Important / Interesting?

Squares on the sides of triangles don't seem much.

But the theorem gives us distances.

Why is this Important / Interesting?

Squares on the sides of triangles don't seem much.

But the theorem gives us distances.

If one point is at $(0, 0)$ and another at (x, y), the theorem gives the distance:

$$r^2 = x^2 + y^2 \qquad \textbf{or} \qquad r = \sqrt{x^2 + y^2}$$

Why is this Important / Interesting?

Squares on the sides of triangles don't seem much.

But the theorem gives us distances.

If one point is at $(0, 0)$ and another at (x, y), the theorem gives the distance:

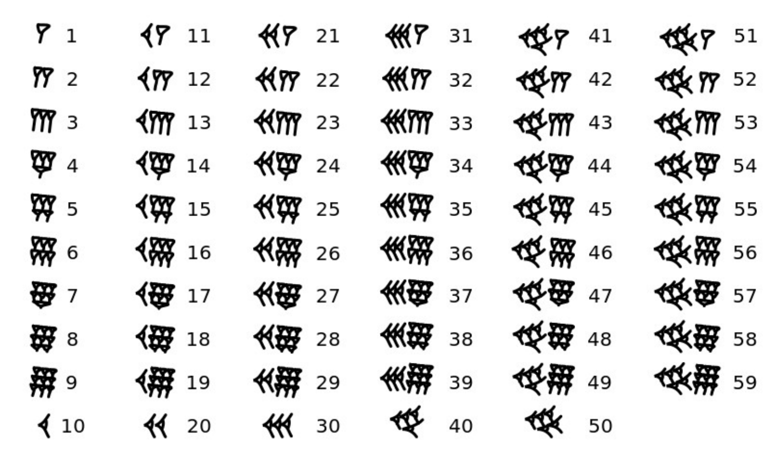
$$r^2 = x^2 + y^2 \qquad \text{or} \qquad r = \sqrt{x^2 + y^2}$$

This tells us about the structure of space.

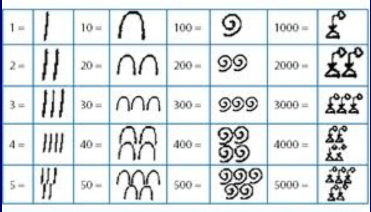
I should expand on this topic. (Example: SAR)

Outline

Babylonian Numerals



Ancient Egyptian Numerals



Ancient Hebrew and Greek Numerals

8	7	6	5	4	3	2	1
ח	ז	ו	ה	ד	ג	ב	א
Chet	Zayin	Vav	Hey	Dalet	Gimmel	Bet	Aleph
ח	ז	ו	ה	ד	ג	ב	א

70	60	50	40	30	20	10	9
ע	ס	נ	מ	ל	כ	י	ט
Ayin	Samekh	Nun	Mem	Lamed	Kaf	Yod	Tet
ע	ס	נ	מ	ל	כ	י	ט

1	α	alpha	10	ι	iota	100	ρ	rho
2	β	beta	20	κ	kappa	200	σ	sigma
3	γ	gamma	30	λ	lambda	300	τ	tau
4	δ	delta	40	μ	mu	400	υ	upsilon
5	ϵ	epsilon	50	ν	nu	500	ϕ	phi
6	ϛ	vau*	60	ξ	xi	600	χ	chi
7	ζ	zeta	70	o	omicron	700	ψ	psi
8	η	eta	80	π	pi	800	ω	omega
9	θ	theta	90	ϙ	koppa*	900	ϡ	sampi

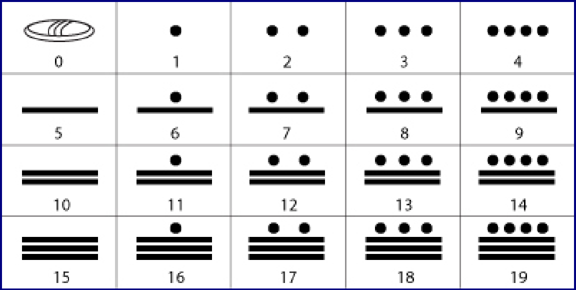
*vau, koppa, and sampi are obsolete characters

Roman Numerals

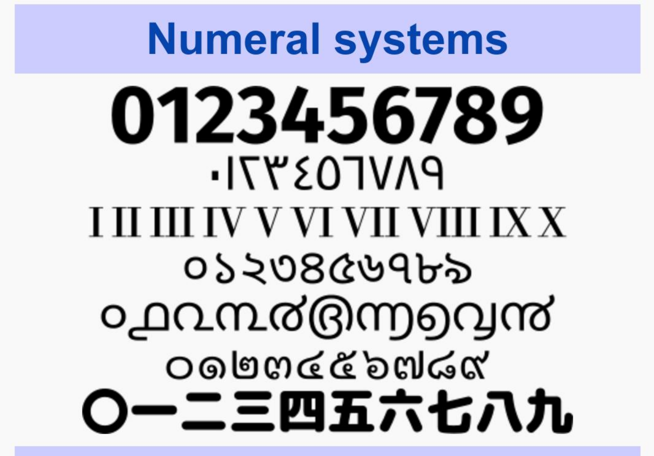
I	1	XXI	21	XLI	41
II	2	XXII	22	XLII	42
III	3	XXIII	23	XLIII	43
IV	4	XXIV	24	XLIV	44
V	5	XXV	25	XLV	45
VI	6	XXVI	26	XLVI	46
VII	7	XXVII	27	XLVII	47
VIII	8	XXVIII	28	XLVIII	48
IX	9	XXIX	29	XLIX	49
X	10	XXX	30	L	50
XI	11	XXXI	31	LI	51
XII	12	XXXII	32	LII	52
XIII	13	XXXIII	33	LIII	53
XIV	14	XXXIV	34	LIV	54
XV	15	XXXV	35	LV	55
XVI	16	XXXVI	36	LVI	56
XVII	17	XXXVII	37	LVII	57
XVIII	18	XXXVIII	38	LVIII	58
XIX	19	XXXIX	39	LIX	59
XX	20	XL	40	LX	60

In order: $M\ D\ C\ L\ X\ V\ I\ = 1666$

Mayan Numerals

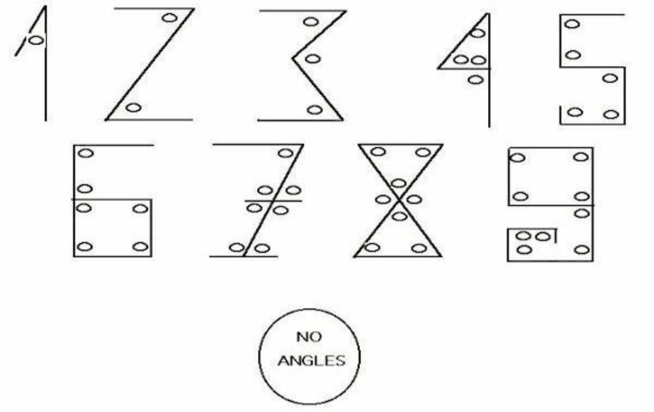


Various Numeral Systems

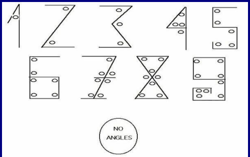


Wikipedia: Hindu-Arabic Numeral System

A Different Angle on Numerals

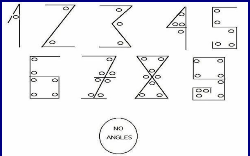


Delightful theory. Almost certainly wrong.



Arguments "for"

1. It is a very simple idea
2. It links numerals to numerical values

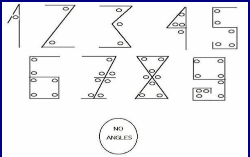


Arguments "for"

1. It is a very simple idea
2. It links numerals to numerical values

Arguments "against"

1. Number forms modified to fit model
2. Complete lack of historical evidence.



Arguments "for"

1. It is a very simple idea
2. It links numerals to numerical values

Arguments "against"

1. Number forms modified to fit model
2. Complete lack of historical evidence.

The great tragedy of science —

the slaying of a beautiful hypothesis by an ugly fact (T H Huxley)

Outline

The Greek Alphabet, Part 3

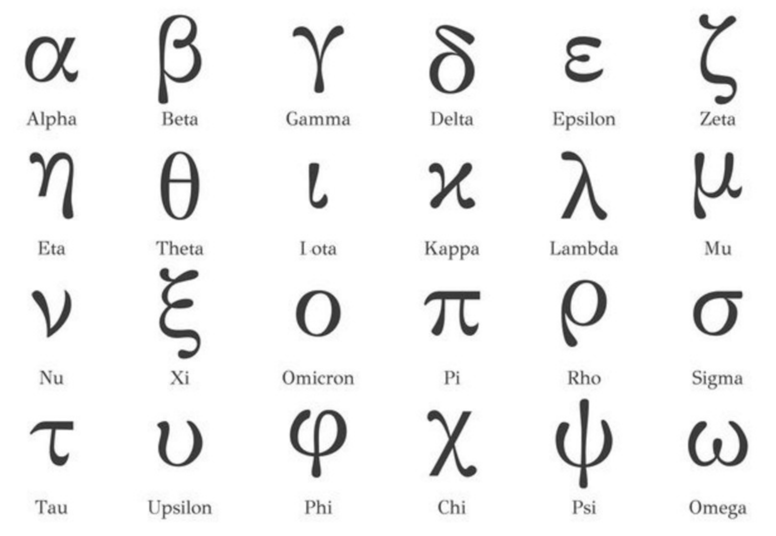


Figure : 24 beautiful letters

The Next Six Letters

We will consider the third group of six letters.

ν	ξ	o	π	ρ	σ
N	Ξ	O	Π	P	Σ

Let us focus first on the small letters
and come back to the big ones later.

$\nu \quad \xi \quad o \quad \pi \quad \rho \quad \sigma$

Nu (ν) is in Planck's formula: $E = h\nu$.
Then ν is the frequency of a photon of light.

Xi (ξ) is the Greek X, as in $\kappa\lambda\iota\mu\alpha\xi$ or $\mathrm{K\Lambda IMAX}$.

Omicron: Think of Oh-Micron, small Oh (not OMG).
Is there a large O, or Oh-Mega ?

Pi (π) is already very familiar to you all.

Rho (ρ) is Greek R, used for density.

Sigma (σ) is the Greek S. At the end of a word it is written ς.

Now we know eighteen letters. We're 75% done!

A Few Greek Words (for practice)

$\kappa\lambda\iota\mu\alpha\xi$
$\delta\rho\alpha\mu\alpha$
$\nu\epsilon\kappa\tau\alpha\rho$
$\kappa\omega\lambda o\nu$
$\kappa o\sigma\mu o\varsigma$

A Few Greek Words (for practice)

$\kappa\lambda\iota\mu\alpha\xi$
$\delta\rho\alpha\mu\alpha$
$\nu\epsilon\kappa\tau\alpha\rho$
$\kappa\omega\lambda o\nu$
$\kappa o\sigma\mu o\varsigma$

Climax: $\kappa\lambda\iota\mu\alpha\xi$
Drama: $\delta\rho\alpha\mu\alpha$
Nectar: $\nu\epsilon\kappa\tau\alpha\rho$
Colon: $\kappa\omega\lambda o\nu$
Cosmos: $\kappa o\sigma\mu o\varsigma$


Κέντρο
Center
Λιμάνι
Port


Μουσειο Ακρόπολης
Acropolis Museum


Αθήνα
Δελφοί
400M


ΤΑΞΙ

End of Greek 103

Outline

A Hierarchy of Numbers

We will introduce a hierarchy of numbers.

Each set is contained in the next one.

They are like a set of nested Russian Dolls:

Matryoshka

The Natural Numbers $\mathbb{N}$

The **counting numbers** were the first to emerge:

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \ \ldots$$

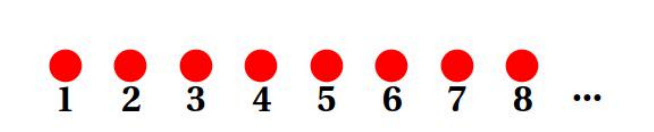
They are also called the **Natural Numbers**.

The Natural Numbers $\mathbb{N}$

The **counting numbers** were the first to emerge:

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \; \ldots$$

They are also called the **Natural Numbers**.



We can arange the natural numbers in a list.

This list is like a **toy computer.**

The Natural Numbers $\mathbb{N}$

The set of natural numbers is denoted $\mathbb{N}$.

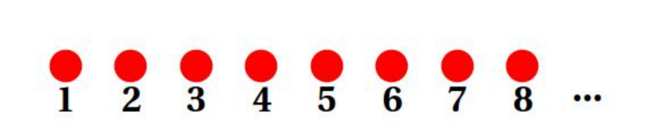
If n is a natural number, we write $n \in \mathbb{N}$.

The Natural Numbers $\mathbb{N}$

The set of natural numbers is denoted $\mathbb{N}$.

If n is a natural number, we write $n \in \mathbb{N}$.

Natural numbers can be added: $4 + 2 = 6 \in \mathbb{N}$



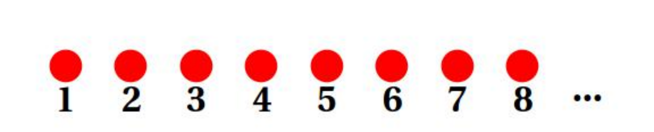
But not always subtracted: $4 - 6 = -2 \notin \mathbb{N}$.

The Natural Numbers $\mathbb{N}$

The set of natural numbers is denoted $\mathbb{N}$.

If n is a natural number, we write $n \in \mathbb{N}$.

Natural numbers can be added: $4 + 2 = 6 \in \mathbb{N}$



But not always subtracted: $4 - 6 = -2 \notin \mathbb{N}$.

To allow for subtraction we have to extend $\mathbb{N}$.

The Integers $\mathbb{Z}$

We extend the counting numbers by adding the negative whole numbers:

$$\ldots \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \ldots$$

The whole numbers are also called the **Integers**.

The Integers $\mathbb{Z}$

We extend the counting numbers by adding the negative whole numbers:

$$\ldots \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \ldots$$

The whole numbers are also called the **Integers**.

The set of integers is denoted $\mathbb{Z}$.

If k is an integer, we write $k \in \mathbb{Z}$.

Clearly,

$$\mathbb{N} \subset \mathbb{Z}$$

Integers can be added and subtracted.

They can also multiplied:

$$6 \times 4 = 24 \in \mathbb{Z} \,.$$

Integers can be added and subtracted.

They can also multiplied:

$$6 \times 4 = 24 \in \mathbb{Z}\,.$$

However, they cannot usually be divided:

$$\frac{6}{4} = 1\tfrac{1}{2} \notin \mathbb{Z}\,.$$

Integers can be added and subtracted.

They can also multiplied:

$$6 \times 4 = 24 \in \mathbb{Z} .$$

However, they cannot usually be divided:

$$\frac{6}{4} = 1\tfrac{1}{2} \notin \mathbb{Z} .$$

To allow for division we have to extend $\mathbb{Z}$.

The Rational Numbers $\mathbb{Q}$

We extend the integers by adding fractions:

$$r = \frac{p}{q} \qquad \text{where } p \text{ and } q \text{ are integers.}$$

These rational numbers are ratios of integers.

The Rational Numbers $\mathbb{Q}$

We extend the integers by adding fractions:

$$r = \frac{p}{q} \qquad \text{where } p \text{ and } q \text{ are integers.}$$

These rational numbers are ratios of integers.

The set of rational numbers is denoted $\mathbb{Q}$.

If r is a rational number, we write $r \in \mathbb{Q}$.

Clearly,

$$\mathbb{Z} \subset \mathbb{Q}$$

With the Rational Numbers, we can:

Add, Subtract, Multiply and Divide

That is, for any $p \in \mathbb{Q}$ and $q \in \mathbb{Q}$

All of $\quad p + q \quad p - q \quad p \times q \quad$ and $\quad p \div q$

are rational numbers.

With the Rational Numbers, we can:

Add, Subtract, Multiply and Divide

That is, for any $p \in \mathbb{Q}$ and $q \in \mathbb{Q}$

All of $\quad p+q \quad p-q \quad p\times q \quad$ and $\quad p\div q$

are rational numbers.

We say that $\mathbb{Q}$ is **closed under addition, subtraction, multiplication and division.**

With the Rational Numbers, we can:

Add, Subtract, Multiply and Divide

That is, for any $p \in \mathbb{Q}$ and $q \in \mathbb{Q}$

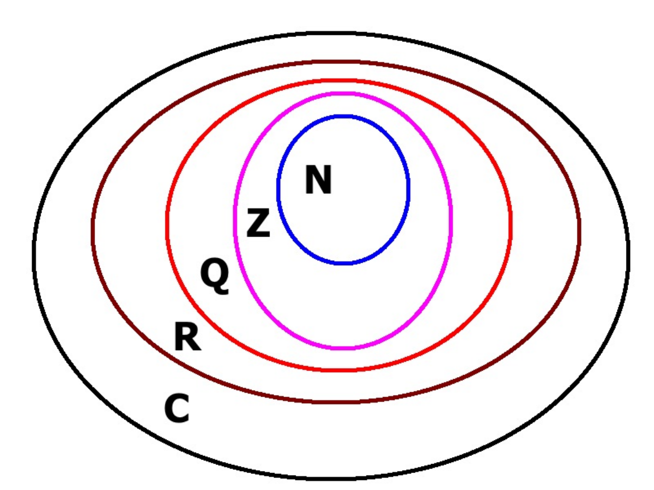
All of $\quad p + q \quad p - q \quad p \times q \quad$ and $\quad p \div q$

are rational numbers.

We say that $\mathbb{Q}$ is **closed under addition, subtraction, multiplication and division.**

But we are not yet finished. $\mathbb{R}$ is yet to come.

The Hierarchy of Numbers



$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

Thank you