AweSums:

The Majesty of Mathematics

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Evening Course, UCD, Autumn 2016

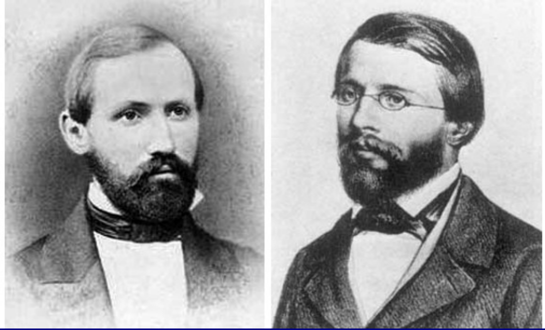
Outline

Outline

AweSums: The Majesty of Maths

Bernhard Riemann (1826-66)

AweSums: The Majesty of Maths



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AweSums: The Majesty of Maths

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We aim to get a flavour of the Riemann Hypothesis.

It involves the zeros of the “Zeta function”:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

So, we need to talk about several new topics:

- What is a function?
- What is an infinite series?
- What about convergence of a series?
- What is a complex variable?

One by one, we will look at all these questions.

Outline

Georg Cantor

Inventor of Set Theory

Born in St. Petersburg, Russia in 1845.

Moved to Germany in 1856 at the age of 11.

His main career was at the University of Halle.

Georg Cantor (1845–1918)

- Invented Set Theory.
- One-to-one Correspondence.
- Infinite and Well-ordered Sets.
- Cardinal and Ordinal Numbers.
- Proved: $\#(\mathbb{R}) > \#(\mathbb{N})$.
- Infinite Hierarchy of Infinities.

Set Theory: Controversy

Cantor was strongly criticized by

- Leopold Kronecker.
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Set theory is a "grave disease" (HP).
Cantor is a "corrupter of youth" (LK).
Set Theory is "nonsense; laughable; wrong!" (LW).

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Set theory is a "grave disease" (HP).
Cantor is a "corrupter of youth" (LK).
Set Theory is "nonsense; laughable; wrong!" (LW).

Adverse criticism like this may well have contributed to Cantor's mental breakdown.

Set Theory: A Difficult Birth

Set Theory brought into prominence several paradoxical results.

Many mathematicians had great difficulty accepting some of the stranger results.

Some of these are still the subject of discussions and disagreement today.

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Many mathematicians had great difficulty accepting some of the stranger results.

Some of these are still the subject of discussions and disagreement today.

To illustrate the difficulty of accepting new ideas, let’s consider the problem of a river flowing uphill.

Describe the blog post “Paddling Uphill”.

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It was so innovative that many mathematicians could not appreciate its fundamental value and importance.

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Gösta Mittag-Leffler was reluctant to publish it in his *Acta Mathematica.* He said the work was "100 years ahead of its time".

David Hilbert said:
"We shall not be expelled from the paradise that Cantor has created for us."

A Passionate Mathematician

In 1874, Cantor married Vally Guttmann.

They had six children. The last one, a son named Rudolph, was born in 1886.

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According to Wikipedia:

“During his honeymoon in the Harz mountains, Cantor spent much time in mathematical discussions with Richard Dedekind.”

[Cantor had met the renowned mathematician Dedekind two years earlier while he was on holiday in Switzerland.]

Distraction: The Simpsons

Several writers of the Simpsons scripts have advanced mathematical training.

They "sneak" jokes into the programmes.

Books on a Shelf

Ten books are arranged on a shelf.
They include an Almanac (A) and a Bible (B).

Suppose A must be to the left of B
(not necssarily beside it).

How many possible arrangements are there?

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The total number of arrangements is 10!.
For half of these, A is to the left of B.

So, answer is $\frac{1}{2}(10 \times 9 \times \cdots \times 1) = \frac{1}{2} \times 10!$ Q.E.D.

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Set Theory I

The concept of set is very general.

Sets are the basic building-blocks of mathematics.

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Examples:

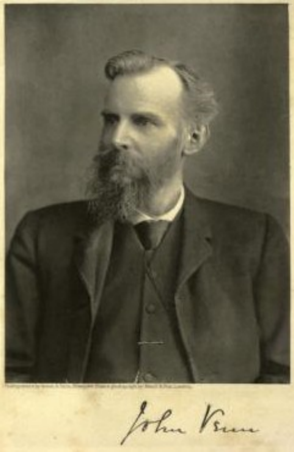
- All the prime numbers, $\mathbb{P}$
- All even numbers: $\mathbb{E} = \{2, 4, 6, 8 \ldots\}$
- All the people in Ireland: See Census returns.
- The colours of the rainbow: {Red, . . . , Violet}.
- Light waves with wavelength $\lambda \in [390 - 700\mathrm{nm}]$

Do You Remember Venn?

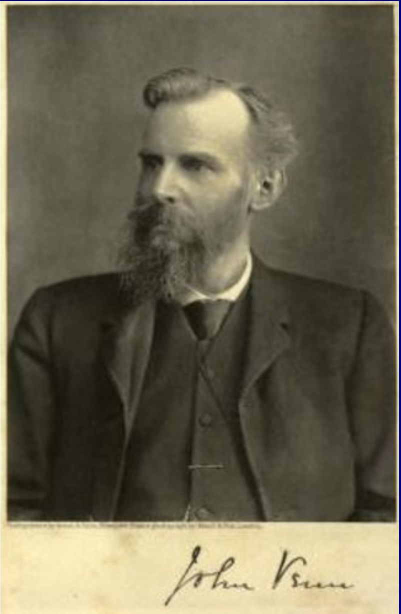
John Venn was a logician and philosopher, born in Hull, Yorkshire in 1834.

He studied at Cambridge University, graduating in 1857 as sixth Wrangler.

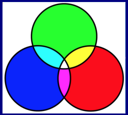
Venn introduced his diagrams in *Symbolic Logic*, a book published in 1881.




JOHN VENN
FELLOW
PRESIDENT


John Venn

Venn Diagrams



Venn diagrams are very valuable for showing elementary properties of sets.

They comprise a number of overlapping circles.

The interior of a circle represents a collection of numbers or objects or perhaps a more abstract set.

The Universe of Discourse

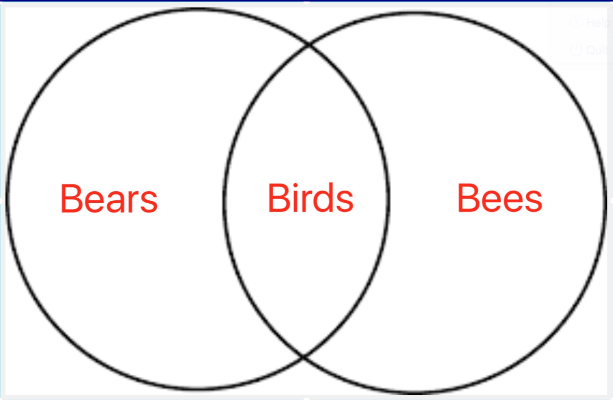
We often draw a rectangle to represent the universe, the set of all objects under current consideration.

For example, suppose we consider all species of animals as the universe.

A rectangle represents this universe.

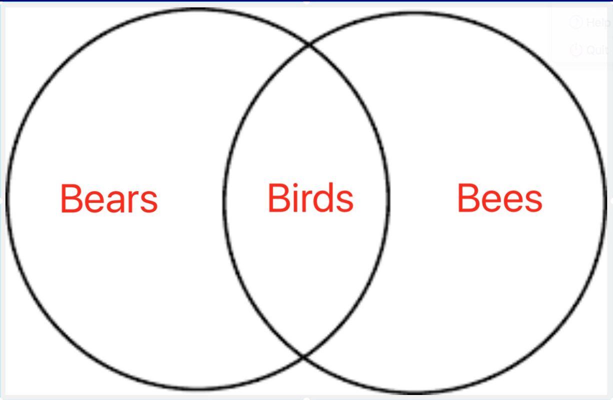
Two circles indicate subsets of animals of two different types.

The Birds and the Bees



Two-legged Animals **Flying Animals**

The Birds and the Bees



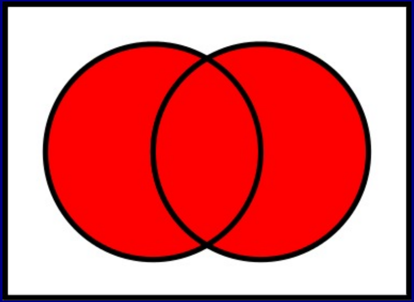
Two-legged Animals Flying Animals

Where do we fit in this diagram?

The Union of Two Sets

The aggregate of two sets is called their union.

Let one set contain all two-legged animals
and the other contain all flying animals.



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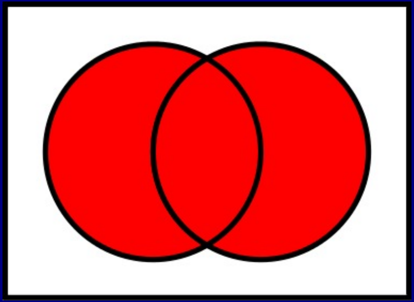
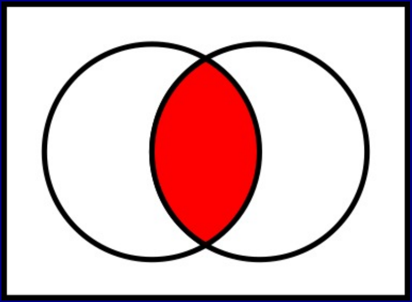


Bears, birds and bees (and we) are in the union.

The Intersection of Two Sets

The elements in both sets make up the intersection.

Let one set contain all two-legged animals and the other contain all flying animals.



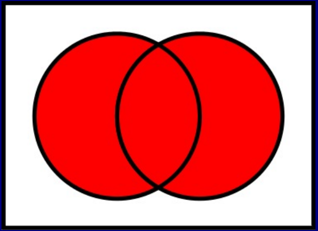
Birds are in the intersection. Bears and bees are not.

The Notation for Union and Intersection

Let A and B be two sets

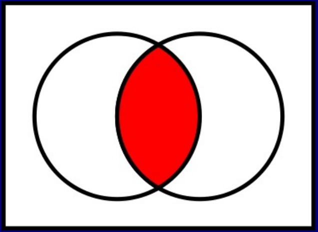
The **union** of the sets is

$$A \cup B$$

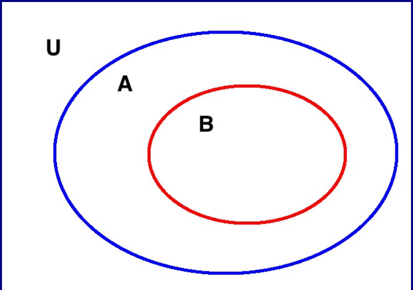


The **intersection** is

$$A \cap B$$



Subset of a Set

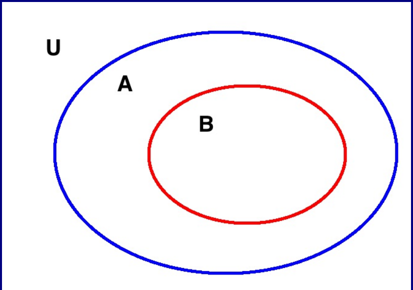


For two sets A and B we write

$$B \subset A \qquad \text{or} \qquad B \subseteq A$$

to denote that B is a subset of A.

Subset of a Set



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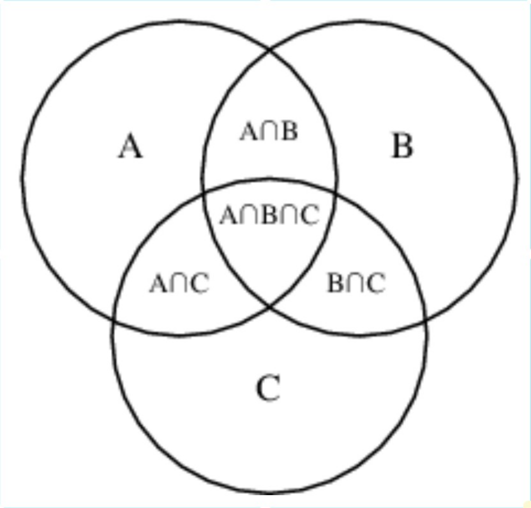
For more on set theory, see website of Claire Wladis

http://www.cwladis.com/math100/Lecture4Sets.htm

Intersections between 3 Sets



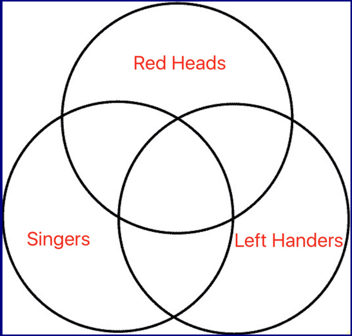
Example: Intersection of 3 Sets

In the diagram the elements of the universe are all the people from Connacht.

Three subsets are shown:

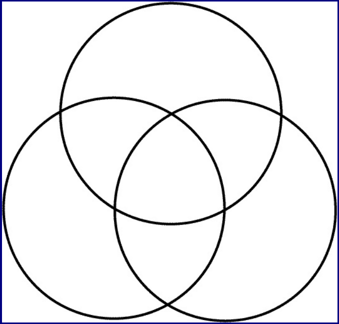
- Red-heads
- Singers
- Left-handers.

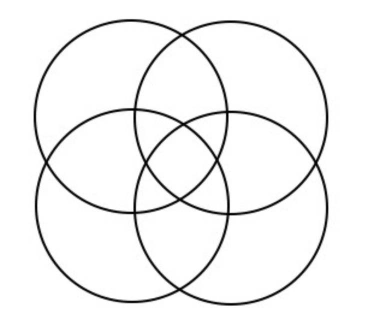
All are from Connacht.



These sets overlap and, indeed, there are some copper-topped, crooning cithogues in Connacht.

Three and Four Sets





8 Domains **14 Domains**

How Many Possibilities?

With just one set A, there are 2 possibilities:

$$x \in A \qquad \text{or} \qquad x \notin A$$

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With two sets, A and B, there are 4 possibilities:

$$(x \in A) \wedge (x \in B) \qquad \textbf{or} \qquad (x \in A) \wedge (x \notin B)$$
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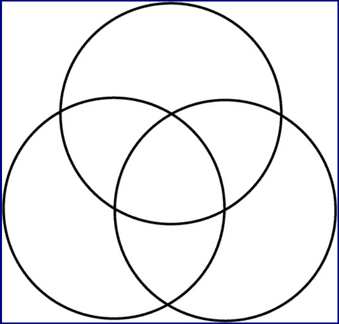
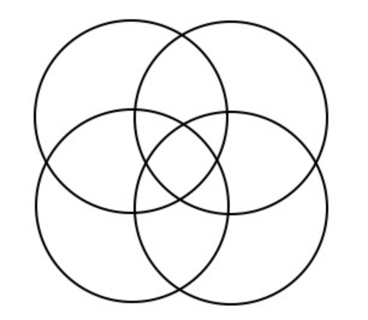
With three sets there are 8 possible conditions.

With four sets there are 16 possible conditions.

Three and Four Sets





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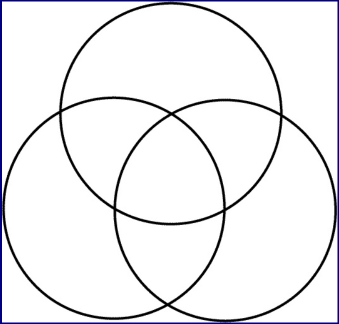

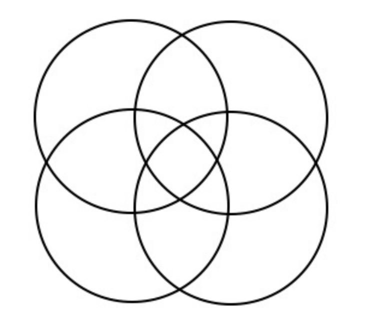

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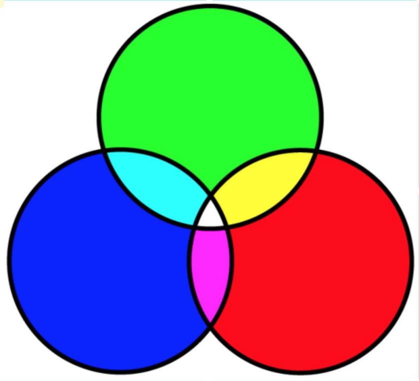
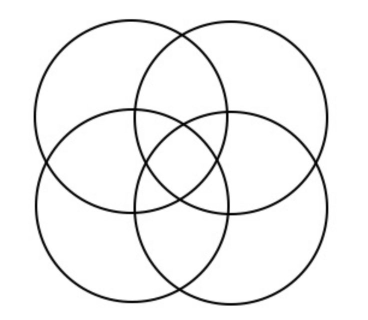
The Intersection of 3 Sets

The three overlapping circles have attained an iconic status, seen in a huge range of contexts.

It is possible to devise Venn diagrams with four sets, but the simplicity of the diagram is lost.



Exercise: Four Set Venn Diagram



Can you modify the 4-set diagram to cover all cases.
(You will not be able to do it with circles only)

Hint: Venn Diagrams for 5 and 7 Sets

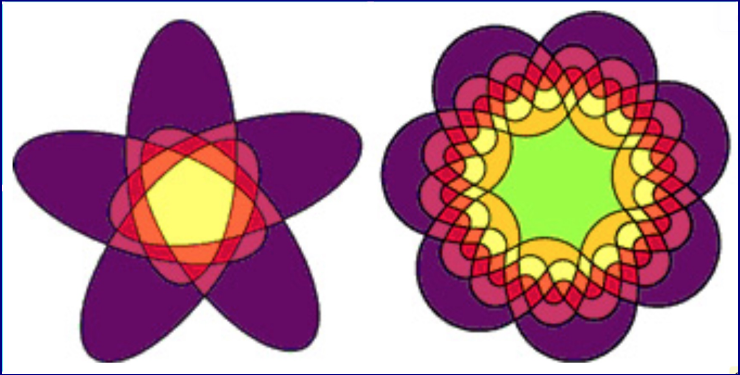
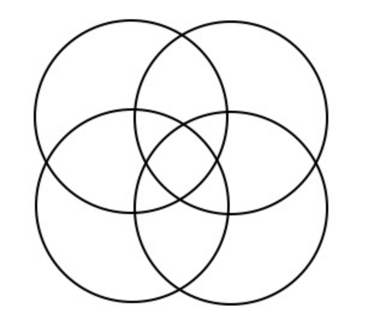


Image from Wolfram MathWorld: Venn Diagram

Solution: Next Week (if you are lucky)



We will find a surprising connection with a Cube

Outline

Interesting Numbers:
Reductio ad Absurdum

Theorem: ***Every number is interesting.***

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Theorem: *Every number is interesting.*

Proof: By *Reductio ad absurdum*.

Give a verbal outline of the proof.

A Ton of Wonders

Article number 100 of the That's Maths series
will appear next Thursday in *The Irish Times*

To celebrate the occassion, I have written a poem

A Ton of Wonders

Here comes a preview, specially for you !!!

The number familiarly known as a ton
Comprises two zeros appended to one.
It holds, in its five score of units, great store
Of marvel and mystery and magic and more.

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Take 1, 2, 3, 4; add them up to make 10.
Then square to one, zero and zero again.
Now square 1, square 7 and double the deuce;
The four squares together one hundred produce.

The number familiarly known as a ton
Comprises two zeros appended to one.
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Take 1, 2, 3, 4; add them up to make 10.
Then square to one, zero and zero again.
Now square 1, square 7 and double the deuce;
The four squares together one hundred produce.

Pythagoras knew, with sides 6, 8 and 10,
A trigon would have a right angle and then,
The squares of the 6 and the 8 being paired,
Make a century for the hypotenuse squared.

The cubes of the first four whole numbers combine
To total one hundred, and not ninety-nine.
With Goldbach to guide us, a century splits
as a sum of two primes, with a half-dozen fits.

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The nine smallest primes up to twenty-and-three
Will sum to precisely a ton, you'll agree.
Now add all odd numbers from 1 to 19:
A sum of a centum again will be seen.

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A number is 'Leyland' if m *to the* n
Plus n *to the* m *gives the number again.*
One hundred is such, as we easily shew,
When m *equals 6, and* n *equals 2.*

One hundred is thrice thirty-three-and-a-third
With many more forms that are much more absurd:
Take a ton from its square: then the iterate root
Brings you back to one hundred without any doubt.

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A ton can be made from irrationals too
And even the powers of transcendents will do:
One hundred is e *plus the fourth power of* π
(albeit this estimate's slightly too high).

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And what of partitions? Of sums there are more
than one-ninety million to make up five score.
This number produces, when broken asunder,
A cornucopia of wealth and of wonder.

$\star \quad \star \quad \star$

Outline

The Greek Alphabet, Part 2

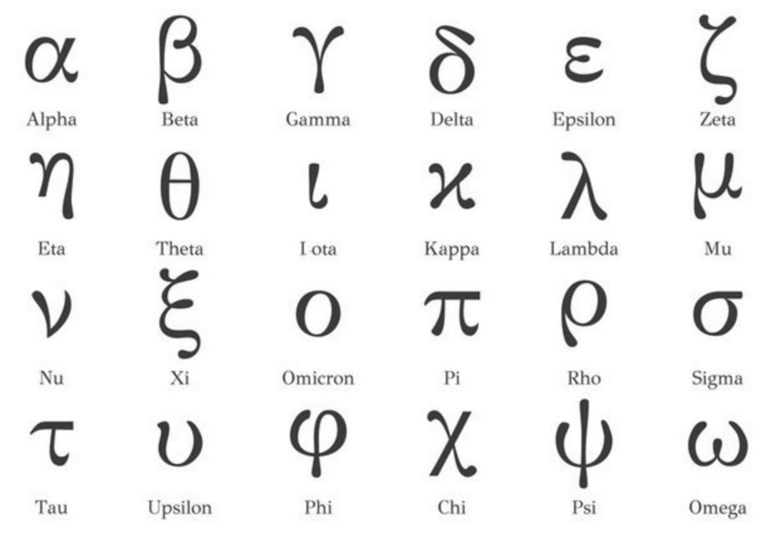

Figure : 24 beautiful letters

The Next Six Letters

We will consider the second group of six letters.

$$\eta \quad \theta \quad \iota \quad \kappa \quad \lambda \quad \mu$$

$$\mathrm{H} \quad \Theta \quad \mathrm{I} \quad \mathrm{K} \quad \Lambda \quad \mathrm{M}$$

Let us focus first on the small letters
and come back to the big ones later.

We already met the Riemann zeta-function; when the signs alternate, it becomes the eta-function:

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \qquad \eta(z) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^z}$$

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The three letters κ, λ, μ are like K, L, M
Also, μ is used for one-millionth: 1μm is a micro-meter.

Now we know the next six letters. We're half way there!

A Few Greek Words (for practice)

$\beta\iota\beta\lambda\iota o$

$\iota\delta\epsilon\alpha$

$\kappa\lambda\iota\mu\alpha\xi$

A Few Greek Words (for practice)

$\beta\iota\beta\lambda\iota o$ | **Book:** $\beta\iota\beta\lambda\iota o$

$\iota\delta\epsilon\alpha$ | **Idea:** $\iota\delta\epsilon\alpha$

$\kappa\lambda\iota\mu\alpha\xi$ | **Climax:** $\kappa\lambda\iota\mu\alpha\xi$

End of Greek 102

Outline

There is No Largest Number

Children often express bemusement at the idea that there is no largest number.

Given any number, 1 can be added to it to give a larger number.

But the implication that there is no limit to this process is perplexing.

The concept of infinity has exercised the greatest minds throughout the history of human thought.

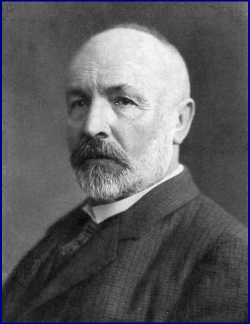
Degrees of Infinity

In the late 19th century, Georg Cantor showed that there are different degrees of infinity.

In fact, there is an infinite hierarchy of infinities.

Cantor brought into prominence several paradoxical results that had a profound impact on the development of logic and of mathematics.

Georg Cantor (1845–1918)



Cantor discovered many remarkable properties of infinite sets.

Cardinality

Finite Sets have a finite number of elements.

Example: The Counties of Ireland form a finite set.

Counties $=$ {Antrim, Armagh, . . . , Wexford, Wicklow}

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For a finite set A, the **cardinality** of A is:

The number of elements in A

One-to-one Correspondence

A particular number, say 5, is associated with all the sets having five elements.

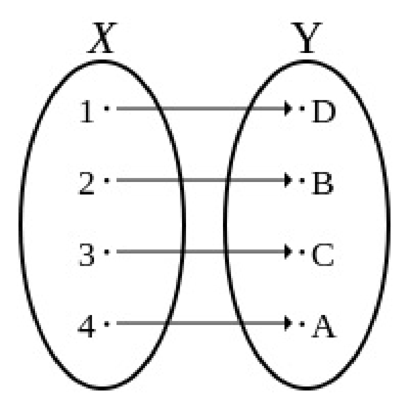
For any two of these sets, we can find a 1-to-1 correspondence between the elements of the two sets.

The number 5 is called the cardinality of these sets.

Generalizing this:

Any two sets are the same size (or cardinality) if there is a 1-to-1 correspondence between them.

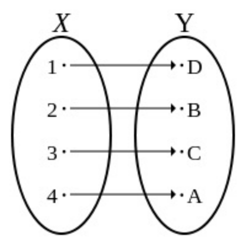
One-to-one Correspondence



Equality of Set Size: 1-1 Correspondence

How do we show that two sets are the same size?

For finite sets, this is straightforward counting.



For infinite sets, we must find a 1-1 correspondence.

Cardinality

The number of elements in a set is called the cardinality of the set.

Cardinality of a set A is written in various ways:

$$|\mathbf{A}| \qquad \|\mathbf{A}\| \qquad \mathrm{card}(\mathbf{A}) \qquad \#(A)$$

For example

$$\#\{\textbf{Irish Counties}\} = 32$$

The Empty Set

We call the set with no elements the empty set.

It is denoted by a special symbol

$$\emptyset = \{ \ \}$$

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$$\#\{ \ \} = 0 \, .$$

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We could have a philosophical discussion about the empty set. Is it related to a perfect vacuum?

The Greeks regarded the vacuum as an impossibility.

The Natural Numbers $\mathbb{N}$

The **counting numbers** (positive whole numbers) are

1 2 3 4 5 6 7 8 . . . .

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The set of natural numbers is denoted $\mathbb{N}$.

This is our first **infinite set.**

We use a special symbol to denote its cardinality:

$$\#(\mathbb{N}) = \aleph_0$$

$$\aleph_0$$

The Power Set

For any set, we can form a new one, the Power Set.

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Suppose the set A has just two elements:

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The power set is

$$\mathrm{P}[\mathbf{A}] = \Big\{\{\ \}, \{a\}, \{b\}, \{a, b\}\Big\}$$

Cantor's Theorem

Cantor's theorem states that, for any set A, the power set of A has a strictly greater cardinality than A itself.

$$\#[\mathrm{P}(\mathbf{A})] > \#[A]$$

This holds for both finite and infinite sets.

It means that, for every cardinal number, there is a greater cardinal number.

One-to-one Correspondence

Now we consider sets are infinite:
take all the natural numbers,

$$\mathbb{N} = \{1, 2, 3, ...\}$$

as one set and all the even numbers

$$\mathbb{E} = \{2, 4, 6, ...\}$$

as the other.

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By associating each number $n \in \mathbb{N}$ with $2n \in \mathbb{E}$, we have a perfect 1-to-1 correspondence.

By Cantor's argument, the two sets are the same size:

$$\#[\mathbb{N}] = \#[\mathbb{E}]$$

Again,

$$\#[\mathbb{N}] = \#[\mathbb{E}]$$

But this is paradoxical: The set of natural numbers contains all the even numbers

$$\mathbb{E} \subset \mathbb{N}$$

and also all the odd ones.

In an intuitive sense, $\mathbb{N}$ is larger than $\mathbb{E}$.

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The same paradoxical result had been deduced by Galileo some 250 years earlier.

Cantor carried these ideas much further:

The set of all the real numbers has a degree of infinity, or cardinality, greater than the counting numbers:

$$\#[\mathbb{R}] > \#[\mathbb{N}]$$

Cantor showed this using an ingenious approach called the diagonal argument.

This is a fascinating technique, but we will not give details here.

Review: Infinities Without Limit

For any set A, the power set P(A) is
the collection of all the subsets of A.

Cantor proved P(A) has cardinality greater than A.

For finite sets, this is obvious;
for infinite ones, it was startling.

The result is now known as Cantor's Theorem,
and Cantor used his diagonal argument in proving it.

He thus developed an entire hierarchy
of transfinite cardinal numbers.

Outline

Set Theory Puzzle

In a small Canadian village, everyone speaks either English or French, or both.

80% of the people speak French

60% of the people speak English

What percentage speak both English and French?

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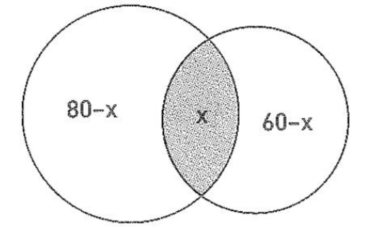
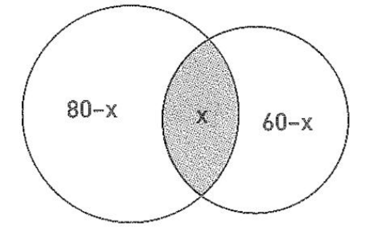
Answer next week!

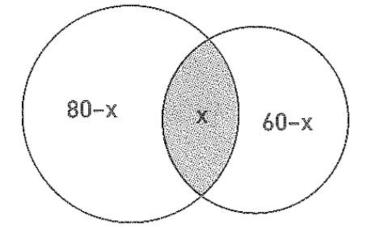

80–x
x
60–x



$$(80 - x) + x + (60 - x) = 100\,.$$

Therefore

$$140 - x = 100 \qquad \textbf{or} \qquad x = 40\,.$$



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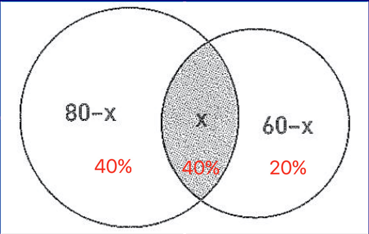

Thank you