What ACM is all about:
2010 – Year of the point sources

Dr Lennon Ó Náraigh [1]

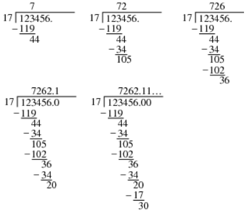
[1] Head of Applied and Computational Mathematics (ACM)
School of Mathematics and Statistics, University College Dublin, Ireland

Introduction

- Three disciplines in school: Applied and computational, Pure, and Statistics.
- Talk about applied and computational maths (ACM).
- Research includes waves, climate, meteorology, general relativity, quantum information, mathematical biology, dynamical systems, **turbulence**, and **mixing**.
- Happily, school possesses world-renowned experts in these fields (not me, unfortunately!).
- I will talk about the topics in bold, and revert to a topic that was current in 2010 – which was less commonly known as **the year of point sources**.

What is research in mathematics?

- First, explain what research in mathematics is – why bother?
- Is it doing big sums all day?



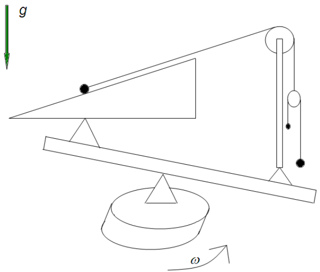
- Obviously not – big questions still to be answered.

What is research in Mathematics?

- Answer – It is playing around with the endless possibilities afforded by abstraction.
- E.g. Combine calculus and complex numbers and obtain **complex analysis**.
- Combine this with **number theory** (nature of prime numbers) and obtain Riemann Hypothesis.
- If you can solve this, you will earn yourself one million dollars!
- Prime numbers of practical importance – cryptography.

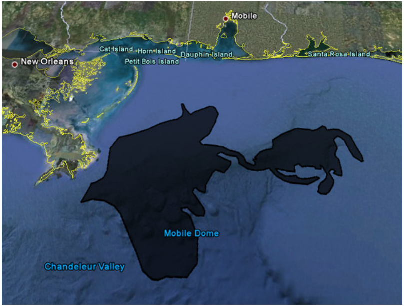
What is applied Mathematics?

- If research in mathematics were about big sums, then surely research in applied mathematics would be like...
- This!



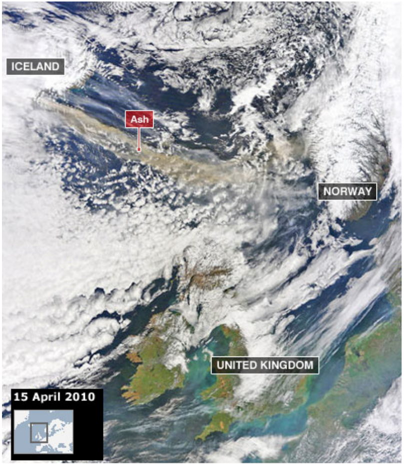
- Not really! Applied mathematics – using any mathematical technique to solve real-world problems.
- Sometimes applied mathematicians have to invent these techniques – two-way exchange between pure and applied.

2010 – Year of point sources



2010 – Year of point sources... continued

(a) 27/03/2010



(b) 15/04/2010

Mathematical modelling

- Concentration field:

$$C = \frac{\text{Amount of bad stuff}}{\text{Unit area}} = \text{function of space and time}$$

- Work in two dimensions because atomsphere and oceans are thin – aspect ratio same as credit card.
- Write down a balance law that relates changes in the concentration:

$$\begin{aligned}\text{Instantaneous change in } C + \text{Stirring of } C &= \text{Diffusion of } C \\ &+ \text{Rate at which matter is put into system} \\ &- \text{Rate at which matter is taken out of system.}\end{aligned}$$

- If matter is neither put into nor taken out of system, this is the statement of conservation of mass – not a very fancy theory!

Mathematical modelling – details

- Instantaneous change in C – $\partial C/\partial t$.
- Stirring – written as $\boldsymbol{u}\cdot\nabla C$. Here $\boldsymbol{u}$ is the velocity of fluid particles. It depends on space and time. Oil spill – ocean currents; volcano – winds.
- Diffusion – written as $D\nabla^2 C$. D – diffusion constant.
- Here ∇ and ∇^2 are derivatives in space. In one dimension, $\nabla \to d/dx$, $\nabla^2 \to d^2/dx^2$.
- Rate at which matter is put into system – localised source.
- Rate at which matter is taken out of system – delocalised sinks. Oil spill – wind shear, biodegradation; volcano – none – wait!! Over short periods, can ignore sinks.

Mathematical modelling – the equation

$$\frac{\partial C}{\partial t} + \boldsymbol{u} \cdot \nabla C = D\nabla^2 C + s(\boldsymbol{x}, t).$$

- This is a **partial differential equation** – if we can find $\boldsymbol{u}$, D, and s, we know $C(\boldsymbol{x}, t)$ for all time.
- D and s can be measured.
- Hence, if we can find $\boldsymbol{u}(\boldsymbol{x}, t)$, we can predict the future concentration (spreading of oil, dangerous flight paths into volcanoes).
- It's a big ask – requires solving the **Navier–Stokes equations** for a long period of time, over a large spatial domain.
- When people do that for NWP, there is a 10-day limit (chaos).
- It's possible to do average-flow and average-concentration predictions (Oil companies use such models).

Averaged equation

- Average concentration C over a period of time and a small length scale: $C \to \langle C \rangle$.
- Same thing for $\langle \boldsymbol{u} \rangle$.
- Obtain average equation:

$$\frac{\partial}{\partial t}\langle C \rangle + \langle \boldsymbol{u} \rangle \cdot \nabla \langle C \rangle = \nabla \cdot (D_{\text{eff}}(\boldsymbol{x}, t) \nabla \langle C \rangle) + \langle s(\boldsymbol{x}, t) \rangle.$$

- Equation is the same except for diffusion, $D \to D_{\text{eff}}(\boldsymbol{x}, t)$.
- Reason for the change is that product of average $\neq$ average of product, e.g.

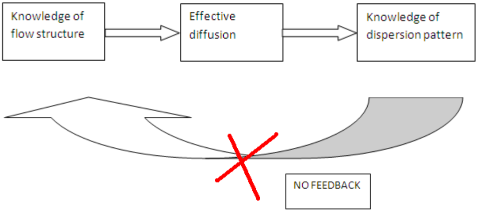
$$a = \{-1, 1\}, \qquad \langle a^2 \rangle = 1, \qquad \langle a \rangle^2 = 0.$$

Averaged equation – issues

- Average equation much easier to solve numerically.
- Small time- and length-scales not resolved – faster on a computer.
- Extra information not ignored – it is bundled into $D_{\mathrm{eff}}(\boldsymbol{x}, t)$.
- We do not know $D_{\mathrm{eff}}(\boldsymbol{x}, t)$ – it has to be **modelled**.
- In other words, we write down a form that takes account of the relevant physics, adjust parameters, and seek agreement with observations.

Averaged equation – flowchart

- ‘Relevant physics’ – depends on the type of flow – knowledge of flow structure needed for solution.
- Also depends on source but this is generic – point source.



- No feedback – can compute effective diffusion once and for all and take it ‘off the shelf’ as needed.
- Dynamics of all future spills / eruptions predicted using this once-off calculation.

Statistical Homogeneous Isotropic Turbulence

- Turbulent flow: flow splits into long-time average, and fluctuations.
- Fluctuations are small in scale and apparently random.
- Randomness – statistics – independent of where you measure or when you start measuring.
- Standard deviation of fluctuations: U_0.
- Fluid particles are 'bumped around' by random motions – very like molecular diffusion.
- Turbulence augments diffusion – hence, **effective diffusion**.
- Source has single (small) lengthscale ℓ – point source.
- Inspired guess:

$$D_{\mathrm{eff}} = U_0 \times \text{form factor in } s = U_0 \ell.$$

Homogeneous Isotropic Turbulence - flowchart

Flow structure:

- Homogeneous
- Isotropic
- Turbulence
- Mean zero, standard deviation U_0
- Source lengthscale l

→

Effective diffusion:

$D_{eff} = U_0$ X Form factor

↓

Knowledge of dispersion pattern: Easy!!

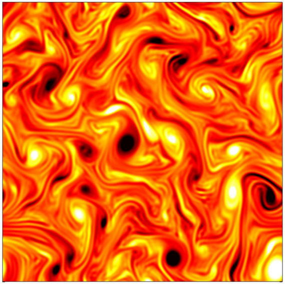
$$\frac{\partial}{\partial t}\langle C\rangle = D_{eff}\nabla^2\langle C\rangle + \langle s\rangle$$

$$\Rightarrow \langle C\rangle(\boldsymbol{x},t) = \sum_{\boldsymbol{k}} e^{ikx - D_{eff}k^2 t}\int_0^t \widetilde{\langle s\rangle}_{\boldsymbol{k}}(t')e^{D_{eff}k^2 t'}dt'$$

$$+\sum_{\boldsymbol{k}} \widetilde{\langle C\rangle}_{\boldsymbol{k}}(t=0)e^{ik.x - D_{eff}k^2 t}$$

Verify

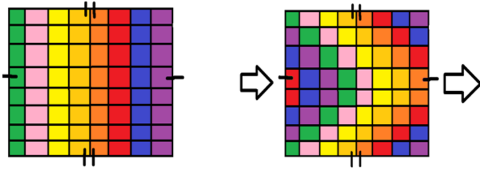
- Before using our 'guess', would like to check that it works – experiments...
- Or numerical simulations (usually cheaper).
- Simulating homogeneous isotropic turbulence – very long time, even in 2D.



The level of vorticity or "swirliness" in two-dimensional turbulence. False color image to show regions of anti-clockwise flow (positive vorticity, bright). Magnitude of vorticity measures shear content – by how much flow 'pulls things apart'.

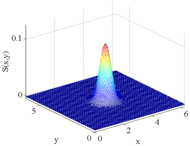
Verify – models

- Instead, use computationally cheap model – random-phase sine flow (RPSF).
- Map: Imagine a chessboard, with each square coloured a different colour. Reconfigure the chessboard so that colours move around on the board. Eventually, the initial configuration is totally scrambled.
- This is exactly what mixing is, and the RPSF does this, and tries to model turbulence (random fluctuations).

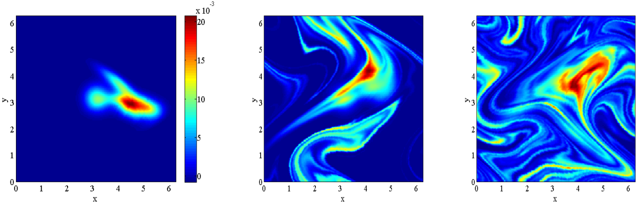


Numerical calculations I

Localised source – close to being a ‘point source’.



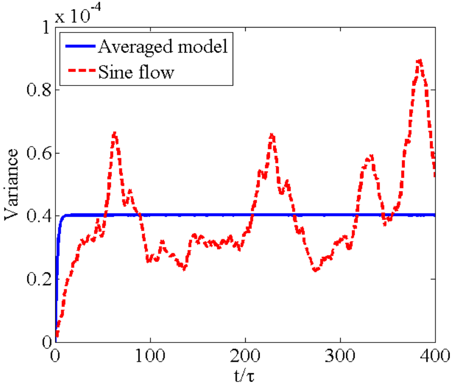
(a)



(b) $t = 5\tau$ (c) $t = 25\tau$ (d) $t = 50\tau$

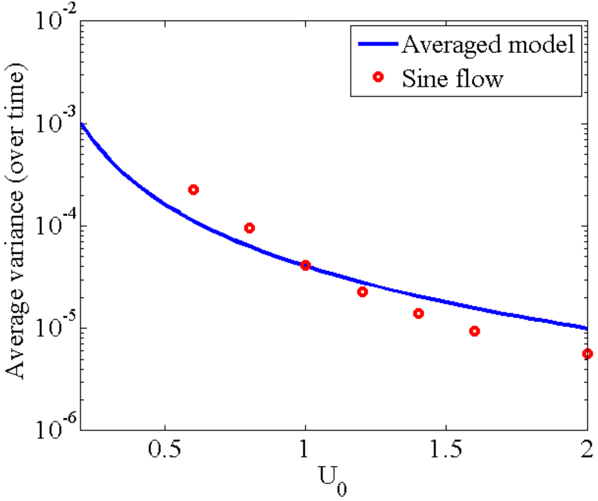
Numerical calculations II

Time-dependence of variance – statistics described by effective-diffusion model.



Not just one lucky datapoint...

Numerical calculations III



Outlook

- Turbulence theories help to compute effective diffusion.
- Calculating dispersion reduced to solving much simpler problem.
- Hence, real-time information about dispersion of pollutants.
- In-house codes from oil companies can fail after 40 days – could we do better??
- Other effects – biodegradation, droplet breakup – future work.

Conclusions

- This talk: Discussed point sources – localised injection of pollutant into a fluid container.
- Studied how this spreads out – ‘dispersion problem’
- Introduced some theories to make numerical simulations easier.
- Relies on notions of turbulence and mixing.
- Applied mathematics research in the school – very contemporary, practical applications.