Question 1.

Start with

$$\frac{\partial}{\partial t}(C-C_0) = \nabla^2 (C - C_0) \text{ on } \Omega,$$

or $$\frac{\partial}{\partial t}\delta C = \nabla^2 \delta C,$$

with initial data to be determined, along with appropriate BCs.

Multiply across by δC and integrate over Ω:

$$\int_\Omega \left(\frac{\partial}{\partial t}\delta C\right)\delta C \, d^n x = \int_\Omega \delta C (\nabla^2 \delta C)\, d^n x$$

or $$\frac{1}{2}\frac{d}{dt}\int_\Omega \delta C^2 \, d^n x = \int_\Omega \delta C (\nabla^2 \delta C) d^n x$$

or $$\frac{1}{2}\frac{d}{dt}\|\delta C\|_2^2 = \int_\Omega \left[\underline{\nabla}\cdot(\delta C \underline{\nabla}\delta C) - |\underline{\nabla}\delta C|^2\right] d^n x$$

$$\overset{\text{divergence}}{=} \int_{\partial\Omega} \delta C(\underline{\nabla}\delta C)\cdot \hat{n}\, dS - \int_\Omega |\nabla \delta C|^2 d^n x$$

We will examine two conditions under which the boundary terms vanish. To obtain a very definite result, these conditions will be much more stringent than those discussed in class. In particular, let us consider first the case where the model diffusion equation and the model Poisson problem are both endowed with homogeneous Neumann II

conditions:

$\hat{n}\cdot\nabla C = 0$ on $\partial\Omega$, diffusion eq$\underline{n}$

$\hat{n}\cdot\nabla C_0 = 0$ on $\partial\Omega$, Poisson eq$\underline{n}$.

Then, $\hat{n}\cdot\nabla\delta C = 0$ on $\partial\Omega$, and the boundary term vanishes.

Thus, we are left with

$$\frac{1}{2}\frac{d}{dt}\|\delta C\|_2^2 = -\int_\Omega |\nabla\delta C|^2 d^n x$$

$$= -\|\nabla\delta C\|_2^2$$

We would now like to use Poincaré's inequality:

$$\|f\|_2 \leq C_\Omega \|\nabla f\|_2$$

for smooth f$\underline{n}$s of mean zero on Ω.

To apply this result, we would need to have $\langle\delta C\rangle = 0$.

Now, by averaging the PDE $\partial_t\delta C = \nabla^2\delta C$, it follows that

$$\langle\delta C\rangle(t) = \langle\delta C\rangle(t=0)$$
$$= \langle C_{init}\rangle - \langle C_0\rangle.$$

Now we know that the sl$\underline{n}$ to the model Poisson problem with Neumann BCs is determined up to a constant only.

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Therefore, we choose to focus on one particular solution in that family of solutions (members of this family differ only by a constant). Indeed, we choose that function C_0 that satisfies

$$\langle C_0 \rangle = \langle C_{init} \rangle ,$$

where $\langle C_{init} \rangle$ is the average of the initial data that are fixed in the formulation of the diffusion problem.

Thus, for the Neumann case, $\langle \delta C \rangle = 0$ for all time, and Poincaré's inequality applies:

$$\|\delta c\|_2 \leq C_\Omega \|\nabla \delta c\|_2$$

$$\|\delta c\|_2^2 \leq C_\Omega^2 \|\nabla \delta c\|_2^2$$

$$-\|\delta c\|_2^2 \geq -C_\Omega^2 \|\nabla \delta c\|_2^2$$

$$-\|\nabla \delta c\|_2^2 \leq -\frac{1}{C_\Omega^2} \|\delta c\|_2^2 .$$

We now revisit the eq$^{\underline{n}}$

$$\frac{1}{2} \frac{d}{dt} \|\delta c\|_2^2 = -\|\nabla \delta c\|_2^2 .$$

Poincaré's inequality now yields

$$\frac{1}{2} \frac{d}{dt} \|\delta c\|_2^2 \leq -\frac{1}{C_\Omega^2} \|\delta c\|_2^2 .$$

Apply Gronwall's inequality and obtain IV

$$\|\delta c\|_2^2(t) \leq e^{-(2/C_\Omega^2)t} \|\delta c\|_2^2(0).$$

Hence, $\lim_{t\to\infty} \|\delta c\|_2^2(t) = 0$

and the model diffusion sol$^{\underline{n}}$ relaxes to a (not "the") model Poisson sol$^{\underline{n}}$, as $t \to \infty$.

Of course, we must also consider the possibility of homogeneous Dirichlet conditions:

$C = 0$ on $\partial\Omega$ for the diffusion eq$^{\underline{n}}$

$C_0 = 0$ on $\partial\Omega$ for the Poisson eq$^{\underline{n}}$,

meaning that $\delta C = 0$ on $\partial\Omega$ for the equation of differences

$$\frac{\partial}{\partial t}\delta C = \nabla^2 \delta C.$$

However, the method applied previously will fail because we do not have à priori knowledge about the behaviour of $\langle \delta C\rangle(t)$; the argument used previously to write V

$$\frac{\partial}{\partial t}\langle c\rangle = \nabla^2\langle c\rangle \Rightarrow \langle c\rangle = \text{const.}$$

~~..~~ relied on Neumann BCs; this argument no longer applies. Moreover, we no longer have control over the value of $\langle c_0\rangle$, since the model Poisson equation $\nabla^2 c_0 + S = 0$ – under Dirichlet conditions – has a uniquely determined solution (MAXIMUM PRINCIPLE). For these reasons, à priori methods fail here, and we must resort again to explicit solutions to determine if the sl^n of the model diffusion equation relaxes to the sl^n of the model Poisson eq^n.

Question 2. VI

Let $f : \Omega \to \mathbb{R}$ have the following properties:

- $\Omega = (0, L_x) \times (0, 1)$
- f is smooth, mean zero
- f periodic in the x-dir$^{\underline{n}}$
- $f(0,z) = f(L_x, z)$.

Then $f(x,z) = \sum_{n=1}^{\infty} \sum_{j=-\infty}^{\infty} f_{jn} \cos\left(\frac{n\pi}{L_z} z\right) e^{i(2\pi/L_x) j}$

$$\frac{\partial f}{\partial x} = \sum_{n=1}^{\infty} \sum_{j=-\infty}^{\infty} f_{jn}\, i\left(\frac{2\pi}{L_x}\right) j \cos\left(\frac{n\pi}{L_z} z\right) e^{i(2\pi/L_x) j}$$

$$\frac{\partial f}{\partial z} = \sum_{n=1}^{\infty} \sum_{j=-\infty}^{\infty} f_{jn} \left(-\frac{\pi n}{L_z}\right) \sin\left(\frac{n\pi z}{L_z}\right) e^{i(2\pi/L_x) j}$$

$$\int_\Omega \left|\frac{\partial f}{\partial x}\right|^2 d^2x \overset{\text{Parseval}}{=} \sum_{n=1}^{\infty} \sum_{j=-\infty}^{\infty} |f_{jn}|^2 \left(\frac{2\pi}{L_x}\right)^2 j^2 \left(\frac{L_x L_z}{2}\right)$$

$$\int_\Omega \left|\frac{\partial f}{\partial z}\right|^2 d^2x = \sum_{n=1}^{\infty} \sum_{j=-\infty}^{\infty} |f_{jn}|^2 \left(\frac{\pi}{L_z}\right)^2 n^2 \left(\frac{L_x L_z}{2}\right)$$

$$\|\nabla f\|_2^2 = \int_\Omega \left[\left|\frac{\partial f}{\partial x}\right|^2 + \left|\frac{\partial f}{\partial z}\right|^2 \right] d^2x$$

$$= \sum_{n=1}^{\infty} \sum_{j=-\infty}^{\infty} |f_{jn}|^2 \frac{L_x L_z}{2} \left[\left(\frac{2\pi}{L_x}\right)^2 j^2 + \left(\frac{\pi}{L_z}\right)^2 n^2 \right]$$

$$\|\nabla f\|_2^2 \overset{j=0,\,n=1}{\geq} \left(\frac{\pi}{L_z}\right)^2 \sum_{n=1}^{\infty} \sum_{j=-\infty}^{\infty} |f_{jn}|^2 \frac{L_x L_z}{2}$$

$$\overset{\text{Parseval}}{=} \left(\frac{\pi}{L_z}\right)^2 \|f\|_2^2 .$$

Hence $\|f\|_2^2 \leq \frac{L_z^2}{\pi^2} \|\nabla f\|_2^2$

Setting $L_z = 1$, we obtain $C_\Omega = 1/\pi$ $\blacksquare$