A differential-equation-based model of the glass ceiling in career progression

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Context of work I

- Group-based discrimination is common in all societies.
- The discrimination can manifest itself in the workplace.
- Certain groups may receive preferential treatment, based on group identity rather than individual merit.
- Important to emphasize that discrimination can take many forms, e.g. Irish / Non-Irish, Working Class / Middle Class, etc.
- Gender-based discrimination has been identified as a current societal challenge, not just in Ireland, but across Europe.

Context of work II

- For definiteness, this talk focuses on gender discrimination, but the methodology that is developed is more widely applicable.
- Specifically, we develop a model based on ordinary differential equations to describe the phenomenon of the **glass ceiling**.

Definition: A glass ceiling is a metaphor used to represent an invisible barrier that keeps a given demographic from rising beyond a certain level in a hierarchy. The metaphor was first coined by feminists in reference to barriers in the careers of high-achieving women – From Wikipedia.

Aim of the talk I

- The talk will introduce a quantitative measure of the glass ceiling – the glass-ceiling index.
- The talk will introduce a framework for modelling the change in the glass ceiling over time.
- The model can be used for one organization or for a whole sector.
- The model is based on Ordinary Differential Equations:
    - Time Evolution
    - Long-time Steady State
    - Model constants - effective 'rate constants'
    - Rate constants define the time to achieve the steady state from given initial conditions – relaxation rate.

Aim of the talk II

- Motivated by topical debates, we will focus on the glass ceiling in Academia.
- There is good summary-data provided by the European Union.
- There are excellent large-scale statistical studies for France, Italy, and Spain – we will use these studies to estimate our rate constants for those countries.

There is not enough data as yet for Ireland – this talk provides a motivation for gathering such data.

Why I am doing this?

Two reasons:

I want to showcase how ACM can be used to do weird and wonderful research. Stephen Strogatz once introduced a mathematical model to model dating. Unfortunately it was just a variant on a predator-prey model. I want to do better and show that differential equations can be used in a novel context to provide real insights into society.

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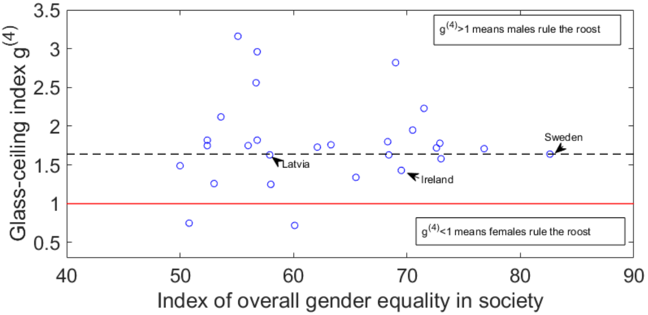
I will use the model to provide a robust description of the **proximate cause** for the glass ceiling in the considered European countries. The **underlying causes** will be left for proper social scientists and psychologists.

The underlying causes are likely to be complicated.



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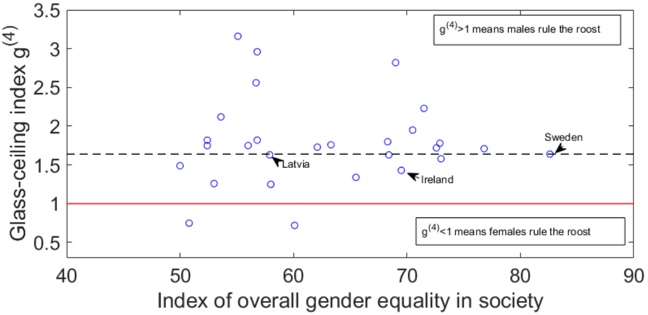

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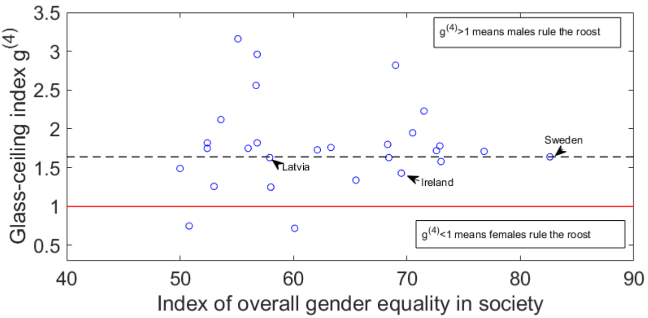

- EU figures 2015 – no correlation between glass-ceiling index (to be defined in talk) and overall level of gender equality in society.
- We can't just 'be more like Sweden' – Sweden has a higher level of overall gender equality than Ireland but fewer women at the top level of academia.
- Causes for gender imbalance at the top of the academic hierarchy are likely to be **complicated** – a mix of social, economic, and cultural factors, that varies country-by-country.

What this talk is not

- To reiterate – I don't know what the factors are, so I won't try to make pronouncements on the social science / psychology literature.
- Instead, I will try to pinpoint the **proximate cause** of the gender imbalance at the top of the hierarchy – this is a matter for modelling and counting, which is definitely within the remit of Applied Mathematics.
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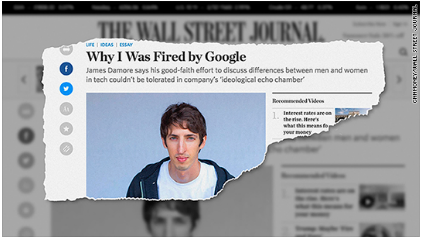

Mathematical Model – Formulation

- Idealized organization, N employees, model N as a fn of time.
- Employees can be categorized in two well-defined, non-overlapping groups, the P-group and the Q-group, with total populations $P_{\text{tot}}(t)$ and $Q_{\text{tot}}(t)$, respectively, such that

$$P_{\text{tot}}(t) + Q_{\text{tot}}(t) = N(t). \tag{1}$$

- P's can be males and Q's can be females, model applies to other dicohotomies as well.
- Idealized organizational hierarchy – two levels – entry level (label 1) and managerial level (label 2). Hence,

$$P_1(t) + P_2(t) = P_{\text{tot}}(t), \qquad Q_1(t) + Q_2(t) = Q_{\text{tot}}(t), \tag{2}$$

Also,

$$P_1(t) + P_2(t) + Q_1(t) + Q_2(t) = N(t). \tag{3}$$

The aim of the model is to describe how members of the P-group and the Q-group progress from the entry level to the managerial level.

Mathematical Model – Assumptions

1. Time is measured in years.
2. The total organizational headcount grows according to $\mathrm{d}N/\mathrm{d}t = \lambda N$, $\lambda = \mathrm{Const.}$
3. Employees leave the organization only through retirements.
4. The organization recruits members of both groups at equal rates. Recruitment is only at the entry level; access to the managerial level is by progression only. No 'demotion' of managers.
5. Employees of the P- and Q-groups retire at equal rates; employees at the different levels in the hierarchy retire at different rates.
6. There is an overall 'crude' retirement rate set by the average length of service.
7. The total number of employees at the managerial level is fixed:

$$\frac{P_2 + Q_2}{N} = \varphi, \qquad \varphi = \mathrm{Constant}, \tag{4}$$

where $0 < \varphi < 1$. Correspondingly, $(P_1 + Q_2)/N = 1 - \varphi$.

Mathematical Model – Warning

- The model is about ‘typical individuals’ and the rate constants are **averages** across entire populations.
- It is therefore important to emphaszie that all statements in talk are about averages and not about individual cases.

The Mathematical Model

Assumptions imply the following ODE model for the P-group:

$$\frac{\mathrm{d}P_1}{\mathrm{d}t} = s - r_1 P_1 - \mu P_1, \tag{5a}$$

$$\frac{\mathrm{d}P_2}{\mathrm{d}t} = \mu P_1 - r_2 P_2. \tag{5b}$$

- s is the source function, depends on time.
- Other coefficients rates, possibly time-dependent:
  - r_1 is the rate at which members of the P-group at the entry level retire.
  - Something similar for r_2.
  - μ is the rate at which members of the P-group at the entry level are promoted to the managerial level.
- Equations (7) are valid for $t > 0$; at $t = 0$, initial conditions apply (obvious notation):

$$P_1(t=0) = P_{10}, \qquad P_2(t=0) = P_{20}, \tag{6}$$

Dimensional Analysis

Dimensional Analysis has played a big role in translating the assumptions into ODEs:

- Source s has has dimensions of $[\text{Number of individuals}][\text{Year}]^{-1}$.
- Rates r_1, r_2, and μ have dimensions of $[\text{Percetage}][\text{Year}]^{-1}$. Hence
  - r_1 is the proportion of a P_1-individuals who retire per year ('retirement rate');
  - μ is the proportion of all P_1-individuals who are promoted to the managerial level, per year ('progression rate').
- The fact that $r_1 \neq 0$ means that some of the members of the P-group at entry level are never promoted to managerial level and spend their whole length of service at the entry level. Note $r_1 \neq r_2$ in general.

The Mathematical Model

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$$\frac{\mathrm{d}P_1}{\mathrm{d}t} = s - r_1 P_1 - \mu P_1, \tag{7a}$$

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Equations for the Q-group

The equations for the Q-group are very similar to those already written down for the P-group:

$$\frac{\mathrm{d}Q_1}{\mathrm{d}t} = s - r_1 Q_1 - \mu' Q_1, \tag{9a}$$

$$\frac{\mathrm{d}Q_2}{\mathrm{d}t} = \mu' Q_1 - r_2 Q_2. \tag{9b}$$

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We begin to characterize the model rigorously now.

Model Characterization – source function

Add up all four model equations:

$$
\begin{aligned}
2s - r_1(P_1 + Q_1) - r_2(P_2 + Q_2) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(P_1 + P_2 + Q_1 + Q_2\right), \\
&= \frac{\mathrm{d}N}{\mathrm{d}t} = \lambda N
\end{aligned}
$$

Hence $N = N_0 \mathrm{e}^{\lambda t}$. Identify the **crude retirement rate** $\hat{r}$, via

$$\hat{r} N = r_1(P_1 + Q_1) + r_2(P_2 + Q_2). \tag{10}$$

Combining the above, we have

$$2s - \hat{r} N = \lambda N, \tag{11}$$

hence

Theorem

The source term is not arbitrary; it is given by

$$s(t) = \tfrac{1}{2}\left(\lambda + \hat{r}\right) N_0 \mathrm{e}^{\lambda t}, \tag{12}$$

Model Characterization – retirement rates

- The source term is a delicate beast – chosen such that the headcount grows at a rate λ, subject to retirements occurring at the crude rate $\hat{r}$.
- The crude retirement rate $\hat{r}$ is known – it is simply the reciprocal of the average length of service T, i.e. $r = 1/T$.
- Similarly, the retirement rate r_2 is known – if T_* is the average time between recruitment and promotion, then $r_2 = (T - T_*)^{-1}$.

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Not all the retirement rates are independent – we now have

Theorem

The retirement rate r_1 is not arbitrary; it is given by

$$r_1 = \frac{\hat{r} - r_2\varphi}{1 - \varphi}. \tag{13}$$

Follows by straightforward messing around with the basic model equations.

Model Characterization – progression rates μ and μ'

Start with $(P_2 + Q_2)/N = \varphi = \mathrm{Const.}$, hence derivative is zero, hence:

$$\frac{\mathrm{d}}{\mathrm{d}t}(P_2 + Q_2) - \frac{1}{N}\frac{\mathrm{d}N}{\mathrm{d}t}(P_2 + Q_2) = 0. \tag{14}$$

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We combine this result with previous equations to obtain

$$\mu P_1 + \mu' Q_1 - r_2(P_2 + Q_2) = \lambda(P_2 + Q_2). \tag{16}$$

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We define a crude progression rate $\hat{\mu}$, such that

$$\hat{\mu}(P_1 + Q_1) = \mu P_1 + \mu' Q_1. \tag{17}$$

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Combining Equations (16) and (17) we have:

Theorem

The crude progression rate $\hat{\mu}$ is not arbitrary; it is given by

$$\hat{\mu} = (r_2 + \lambda)\left(\frac{\varphi}{1 - \varphi}\right). \tag{18}$$

Different progression rates – mathematical formulation

The value $\hat{\mu}$ is fixed, so write μ and μ' more succinctly:

$$\mu' = k\mu,$$

where k is a non-negative constant. Here,

- $k < 1$ indicates a preference for the P-group in the organization's promotion system;
- $k > 1$ indicates a preference for the Q-group.

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In view of previous results, we have

$$\mu = \hat{\mu}\left(\frac{P_1 + Q_1}{P_1 + kQ_1}\right).$$

This can be written succinctly as

$$\mu = \hat{\mu}\Psi(P_1, Q_1), \qquad \Psi = \frac{P_1 + Q_1}{P_1 + kQ_1}.$$

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The function Ψ is **nonlinear** and **homogeneous** in each of its variables, i.e. $\Psi(xP_1, xQ_1) = \Psi(P_1, Q_1)$, for all $x \neq 0$.

Summary of progress so far

We now summarize the model equations in once place:

$$\frac{\mathrm{d}P_1}{\mathrm{d}t} = s(t) - r_1 P_1 - \hat{\mu}\Psi(P_1, Q_1)P_1, \tag{19a}$$

$$\frac{\mathrm{d}P_2}{\mathrm{d}t} = \hat{\mu}\Psi(P_1, Q_1)P_1 - r_2 P_2, \tag{19b}$$

$$\frac{\mathrm{d}Q_1}{\mathrm{d}t} = = s(t) - r_1 Q_1 - \hat{\mu}k\Psi(P_1, Q_1)Q_1, \tag{19c}$$

$$\frac{\mathrm{d}Q_2}{\mathrm{d}t} = \hat{\mu}k\Psi(P_1, Q_1)Q_1 - r_2 Q_2, \tag{19d}$$

where

$$s(t) = \tfrac{1}{2}\left(\lambda + \hat{r}\right) N(t), \qquad N(t) = N_0 \mathrm{e}^{\lambda t}, \tag{19e}$$

$$\hat{\mu} = (r_2 + \lambda)\left(\frac{\varphi}{1-\varphi}\right), \tag{19f}$$

$$r_1 = \frac{\hat{r} - r_2\varphi}{1-\varphi}. \tag{19g}$$

Apart from the initial conditions, the model contains only four parameters that need to be supplied – $\hat{r}$, r_2, k, and φ.

Scaling I

- For the structure of the organization in terms of the P- and Q-groups, what is of interest is not the headcounts $P_1, \ldots, Q_2$ but rather the proportion of individuals at a given career level.
- Hence, introduce

$$p_1 = P_1/N, \qquad p_2 = P_2/N, \qquad q_1 = Q_1/N, \qquad q_2 = Q_2/N. \tag{20}$$

Scaling I

Theorem

Given Equation (19) the scaled variables in Equation (20) satisfy the following ODEs:

$$\frac{\mathrm{d}p_1}{\mathrm{d}t} = s_0 - (r_1 + \lambda)p_1 - \hat{\mu}\Psi(p_1, q_1)p_1, \tag{21a}$$

$$\frac{\mathrm{d}p_2}{\mathrm{d}t} = \hat{\mu}\Psi(p_1, q_1)p_1 - (r_2 + \lambda)p_2, \tag{21b}$$

$$\frac{\mathrm{d}q_1}{\mathrm{d}t} = s_0 - (r_1 + \lambda)q_1 - \hat{\mu}k\Psi(p_1, q_1)q_1, \tag{21c}$$

$$\frac{\mathrm{d}q_2}{\mathrm{d}t} = \hat{\mu}k\Psi(p_1, q_1)q_1 - (r_2 + \lambda)q_2, \tag{21d}$$

where $s_0 = s(t)/N(t) = (\lambda + \hat{r})/2$.

- The proof follows by direct computation; the homogeneity of the function Ψ is a key part of the computation.
- In the remainder of the paper we work with the scaled model (21).

Steady State $\mathrm{d}/\mathrm{d}t = 0$

We examine steady-state solutions of Equation (21) obtained by setting the time derivatives on the left-hand side equal to zero. Hence,

$$s_0 = (r_1 + \lambda)p_1 + \hat{\mu}\Psi p_1, \qquad \hat{\mu}\Psi p_1 = (r_2 + \lambda)p_2, \tag{22a}$$

$$s_0 = (r_1 + \lambda)q_1 + k\hat{\mu}\Psi q_1 \qquad \hat{\mu}k\Psi q_1 = (r_2 + \lambda)q_2. \tag{22b}$$

Combining these equations gives

$$\frac{s_0 - (r_2 + \lambda)q_2}{s_0 - (r_2 + \lambda)p_2} = \frac{q_1}{p_1}, \qquad k\frac{q_1}{p_1} = \frac{q_2}{p_2} \tag{22c}$$

Steady state – anomalous case $k = 0$

- The case $k = 0$ is anomalous, and corresponds to the steady state $q_2 = 0$.
- In this case, Equations (22) reduce to

$$\begin{aligned} s_0 &= (r_1 + \lambda)p_1 + \hat{\mu}\Psi p_1, \\ s_0 &= (r_1 + \lambda)q_1. \end{aligned}$$

- Implies $q_1 = s_0/(r_1 + \lambda)$.
- But $p_1 + q_1 = 1 - \varphi$, hence

$$p_1 = 1 - \varphi - \frac{s_0}{r_1 + \lambda}.$$

- p_1 is a population, we require $p_1 \geq 0$, hence

$$\varphi \leq \frac{\hat{r} + \lambda - s_0}{r_2 + \lambda}. \tag{23}$$

This is a sufficient condition such that $p_1 \geq 0$ at the steady state; indeed, this can be assumed to be a general condition to avoid a population crash where $p_1 \to 0$ in finite time. As such, in the remainder of this work, we assume that Equation (23) holds.

Steady State – Special Case $k = 1$ and General Solution

- When $k = 1$ the p- and q-populations are symmetric, so $p_2 = q_2 = \varphi/2$, and $p_1 = q_1 = (1 - \varphi)/2$ in the steady state.
- Otherwise, there is a general solution in terms of a parameter x:

$$x = q_2/p_2 \implies xp_2 = q_2, \qquad p_2 + q_2 = \varphi \implies p_2 = \varphi/(1 + x). \tag{24}$$

- In this way, the algebraic steady-state equations can be reduced to a quadratic in x, with sensible solution

$$x = -\tfrac{1}{2}(k-1)\left[\frac{\varphi(r_2+\lambda)}{s_0} - 1\right] + \tfrac{1}{2}\sqrt{(k-1)^2\left[\frac{\varphi(r_2+\lambda)}{s_0} - 1\right]^2 + 4k}. \tag{25}$$

- Full steady-state solution parametrized by x (hence, k): fixed by the parameter x – summarized here as follows:

$$(p_{1*}, p_{2*}, q_{1*}, q_{2*}) = \left(\frac{1-\varphi}{1+(x/k)}, \frac{\varphi}{1+x}, \frac{x}{k}\frac{1-\varphi}{1+(x/k)}, \frac{x\varphi}{1+x}\right). \tag{26}$$

Glass-ceiling index

- We introduce the **glass-ceiling index**:

$$
\begin{aligned}
g(t) &= \frac{\text{Proportion of organization made up by } Q\text{-group}}{\text{Proportion of managerial level made up by } Q\text{-group}}, \\
&= \frac{(Q_1 + Q_2)/N}{Q_2/(Q_2 + P_2)}, \\
&= (q_1 + q_2)\left(\frac{q_2 + p_2}{q_2}\right). \qquad (27)
\end{aligned}
$$

- Hence,

$$g(t) = \varphi\left(1 + \frac{q_1}{q_2}\right). \qquad (28)$$

- Correspondingly, we introduce $g_* = \lim_{t\to\infty} g(t)$. Hence,

$$g_* = \varphi\left(1 + \frac{1-\varphi}{\varphi}\frac{1+x}{k+x}\right). \qquad (29)$$

- g_* gives a nice way of visualizing the steady-state solutions.

Glass-Ceiling Index – Plot

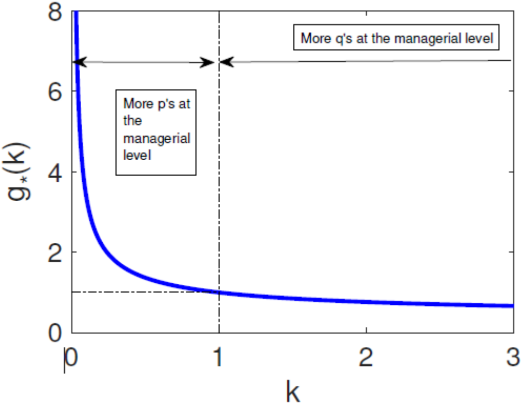

Figure 1. Glass-ceiling index g_* as a function of the asymmetry parameter k, as given by Equation ([illegible]). Model parameters – for illustration purposes only: $\lambda = 0$ (steady-state headcount), $\hat{r} = (1/35)\,[\text{Years}]^{-1}$, $r_2 = (1/15)\,[\text{Years}]^{-1}$, $\varphi = 0.245$

Dyanmics – Exact Model Solutions

When $k = 1$ the P and Q ODEs decouple and become linear. Then they are first-order linear ODEs, which can be solved via the IF technique. FWIW, we obtain

$$p_1(t) = \left[p_1(0) - \frac{s_0}{r_1 + \hat{\mu} + \lambda}\right] \mathrm{e}^{-(r_1+\hat{\mu}+\lambda)t} + \frac{s_0}{r_1 + \hat{\mu} + \lambda}. \tag{30}$$

and

$$p_2(t) = \mathrm{e}^{-(r_2+\lambda)t} \times \left\{p_2(0) - \hat{\mu}\frac{s_0}{r_1 + \hat{\mu} + \lambda}\frac{1}{r_2 + \lambda} - \hat{\mu}\left[p_1(0) - \frac{1}{r_1 + \hat{\mu} + \lambda}\right]\frac{1}{(r_2 - r_1) - \hat{\mu}}\right\} + \hat{\mu}\frac{\left[p_1(0) - \frac{1}{r_1+\hat{\mu}+\lambda}\right]\mathrm{e}^{-(r_1+\hat{\mu}+\lambda)t}}{(r_2 - r_1) - \hat{\mu}} + \frac{\hat{\mu}}{r_2 + \lambda}\frac{s_0}{r_1 + \hat{\mu} + \lambda}. \tag{31}$$

The q-solutions are a copy. Important thing here is the **time constants** which govern how fast the solution relaxes to the steady state and forgets the initial conditions.

The Time Constants

- From the exact solutions we identify the time constants / decay rates

$$\tau_1 = (r_1 + \hat{\mu} + \lambda)^{-1}, \qquad \tau_2 = (r_2 + \lambda)^{-1},$$

- Attenuation of ICs is only as fast as the longest timescale of τ_1 and τ_2.
- Hence, for $\lambda \geq 0$, we identify an overall attenuation rate

$$\tau_* = \max(\tau_1, \tau_2) = (r_2 + \lambda)^{-1}. \qquad (32)$$

- We work with $\lambda \geq 0$ (strange quirk of universities!).

From the relation $\tau_* = \max(\tau_1, \tau_2) = (r_2 + \lambda)^{-1}$ it is clear that both the retirement rate r_2 and a headcount growth rate $\lambda \geq 0$ act together to attenuate the initial conditions and to hasten the onset of the steady state. As such, the onset of the steady state can be hastened by increasing either r_2 or λ.

Numerical Solution of ODE model, $k = 1$

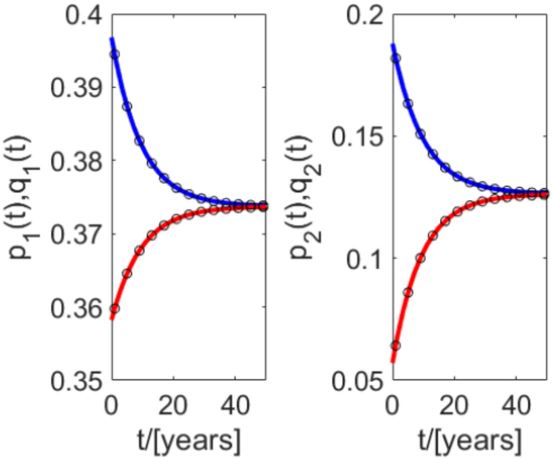

Figure 2. Sample numerical solution of Equation ([illegible]) with $k = 1$. The initial conditions can be read from the graph. Other parameters: $\hat{r} = 1/35\,[\text{Years}]^{-1}$, $r_2 = 1/15\,[\text{Years}]^{-1}$. A headcount growth $\lambda = 1/20\,[\text{Years}]^{-1}$ is also assumed. Numerical method: solid lines. Analytical solution (valid for $k = 1$): circles.

With $k = 1$, you can bring g close to 1 very fast by increasing λ.

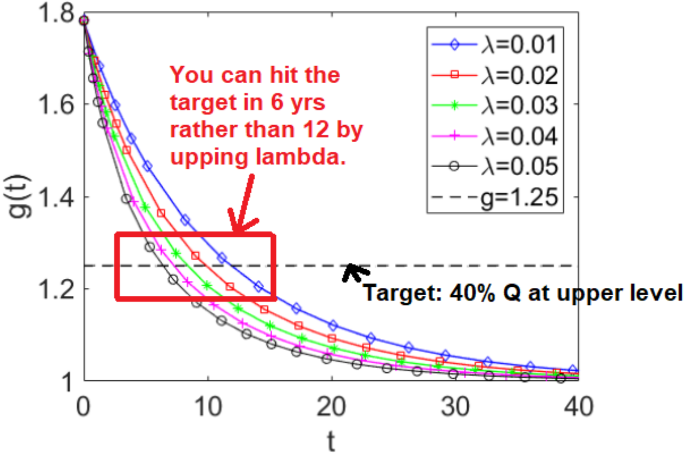

Figure 3. The glass-ceiling index $g(t)$ as a function of time for various values of the headcount growth rate λ. All other parameter values as in Figure 2.

Decomposition of Progression Rate

- For the P-group, we identify

$$\begin{aligned}\mu &= \left[\begin{array}{c}\text{Number of } P\text{-individuals moving to the managerial level,}\\ \text{as a proportion of all } P\text{-individuals at the entry level}\end{array}\right] \Big/ [\text{Year}]\\ &= \left[\begin{array}{c}\text{Number of } P\text{-individuals under consideration for promotion,}\\ \text{as a proportion of all } P\text{-individuals at the entry level}\end{array}\right]\\ &\quad \times [\text{Success rate of the } P\text{-individuals in the promotion system}] \Big/ [\text{Year}]\\ &= \nu \times \sigma,\end{aligned}$$

- We similarly write $\mu' = \nu' \times \sigma'$ (in an obvious notation).

Classification of Bias

We classify the promotion system as follows:

- Supply-side effect: $\sigma = \sigma'$, $\nu \neq \nu'$.
- In-competition effect ('bias'): $\nu = \nu'$, $\sigma \neq \sigma'$.
- Multiple effects: $\nu \neq \nu'$ and $\sigma \neq \sigma'$.
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Hence, a difference in the progression rates $\mu \neq \mu'$ may be the result of one of the following distinct effects:

- Supply-side effect: $\sigma = \sigma'$, $\nu \neq \nu'$, hence $\mu \neq \mu'$.
- In-competition effect ('bias'): $\nu = \nu'$, $\sigma \neq \sigma'$, hence $\mu \neq \mu'$.
- Multiple effects: $\nu \neq \nu'$ and $\sigma \neq \sigma'$, such that $\nu\sigma \neq \nu'\sigma'$.

Cascade Model

Definition: The cascade model stipulates that the proportion of Ps and Qs to be recruited or promoted to a certain level is based on the proportion of each at the career level directly below.

HEA has asked Irish Universities to look into this – universities have agreed to 'targets' based on the model.

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It is easy to show:

Theorem

The cascade model requires that $k = 1$.

Cascade Model – Implications

The cascade model is about equality of μ and μ'. However, this can be achieved in one of three ways, not all necessarily 'benign':

- Supply-side adjustment: $\sigma' \neq \sigma$ – adjust $\nu = \sigma'\nu'/\sigma$, such that $\mu = \mu'$.
- In-competition adjustment: $\nu' \neq \nu$ – adjust $\sigma = \sigma'\nu'/\nu$, such that $\mu = \mu'$.
- Symmetry: $\nu = \nu'$, $\sigma = \sigma'$, such that $\mu = \mu'$.

In particular,

Corollary

If the promotion system exhibits a supply-side bias $\nu' \neq \nu$, then implementation of the cascade model requires the presence of a compensatory in-competition bias.

4-level model

Two-level model not realisitc. Universities throughout Europe have standardized academic career structure into 4 levels – these map nicely onto Irish system (old money) of Lecturer (D) / Senior Lecturer (C) / Associate Professor (B) / Full Professor (A). An extension of our model to allow for four levels is straightforward:

$$\frac{\mathrm{d}P_D}{\mathrm{d}t} = S_P(t) - r_D P_D - \mu_D P_D, \tag{33a}$$

$$\frac{\mathrm{d}P_C}{\mathrm{d}t} = \mu_D P_D - r_C P_C - \mu_C P_C, \tag{33b}$$

$$\frac{\mathrm{d}P_B}{\mathrm{d}t} = \mu_C P_C - r_B P_B - \mu_B P_B, \tag{33c}$$

$$\frac{\mathrm{d}P_A}{\mathrm{d}t} = \mu_B P_B - r_A P_A. \tag{33d}$$

Similarly, for the Q-group (retirement rates the same, progression rates have a prime).

4-level model – glass-ceiling index

Use the EU definition of the glass-ceiling index for the 4-level career structure:

$$g^{(4)}(t) = \frac{Q_A + Q_B + Q_C}{Q_A + Q_B + Q_C + P_A + P_B + P_C} \Big/ \frac{Q_A}{Q_A + P_A} \tag{34}$$

- This helps us to pinpoint the bottlenecks in the attainment of $g^{(4)} = 1$.
- First place to look (limiting factor) – progressions from B to A.
- Standardized data available for for Italy, France, and Spain.
- Reason – they organize national central competitions to determine promotion to the highest academic grade, hence data available, hence, statistically robust can conclusions to be drawn for these countries.
- Summarize France and Spain for brevity – Italy is similar to Spain.

Spain I

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- A regression model is used to determine how the probability p of success in the competition depends on the gender of applicants, as well as other applicant attributes (age, academic productivity, etc.).
- By random selection, some evaluation committees have an all-male composition – this facilitates a 'natural experiment' whereby the effect of the committee composition on promotion prospects can be studied.

Spain II

- For academics in the competition, p depends on gender – males have on average a higher probability of success. The difference is quantified (below).
- The difference goes away when candidates are assessed by mixed-gender panels.
- The regression analysis can be used to estimate the parameters of our own model, σ, ν, etc – we have backed out:

$$\frac{\Delta(\sigma\nu)}{\sigma\nu} = \frac{\Delta\nu}{\nu} + \frac{\Delta\sigma}{\sigma}. \tag{35}$$

with $\Delta\nu/\nu = (1.36 - 1)/1.36 = 0.26$ and $\Delta\sigma/\sigma = (1.14 - 1)/1.14 = 0.12$. Hence,

- The effect of $\nu \neq \nu'$ contributes twice as strongly as the effect of $\sigma \neq \sigma'$
- The system is asymmetric between males and females mostly because of supply-side bias, but in-competition bias plays a role also.

France I (Economics)

- Bosquet et al. (2014) examine the French academic promotion system in Economics between 1991-2008.
- Promotion (in all subjects) is based on a national competition (*concours*). Candidates are evaluated by an evaluation committee.
- There are two academic career tracks: the universities, and the research institutes (CNRS). The *concours* for each career track has its own characteristics. The paper compares the outcomes of the two types of *concours*.
- The data presented in the study consists of academics who applied for promotion, and those who did not. The study therefore distinguishes between these two groups, and introduces a probability $p(S)$ of success for a candidate, conditional on his/her having applied for promotion.

France II (Economics)

- A regression model for probability of in-competition success $p(S)$ is constructed. This shows there are differences in $p(S)$ for males and females, but they are not statistically significant.
- The main reason for the gender asymmetry at the top of the hierarchy is in the difference in the proportion of males and females who enter the promotion competitions, i.e. a supply-side effect.
- Hence, $\sigma = \sigma'$ but $\nu \neq \nu'$.

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- The playing field is level (Fr) and can be levelled (Sp, It) but not all players are showing up for the match.
- Understanding this effect is about getting at the underlying cause of the glass ceiling. The causes could be benign or malign (e.g. 'structural inequality'). Beyond the scope of this research (just counting) to say which.