Understanding turbulence with mathematical models: from 2x2 matrices to 1,000 CPU simulations

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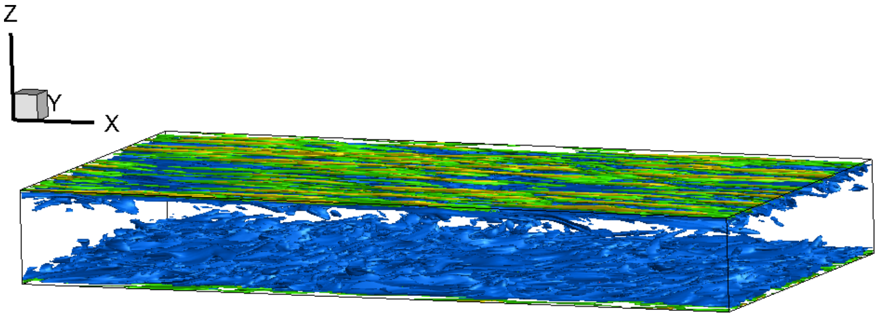
What is turbulence?

Turbulence in a fluid is characterized by chaotic motion on many lengthscales.



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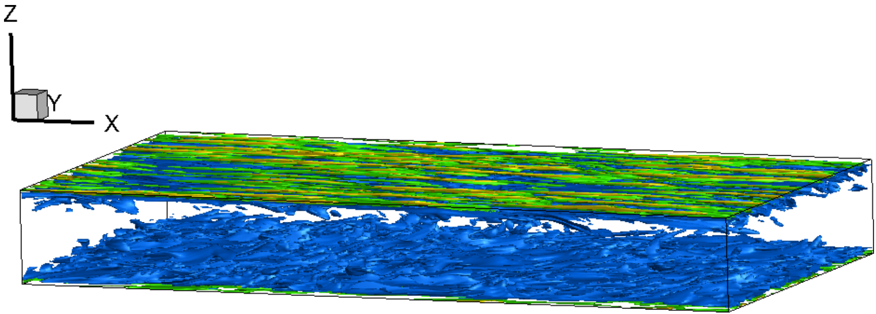

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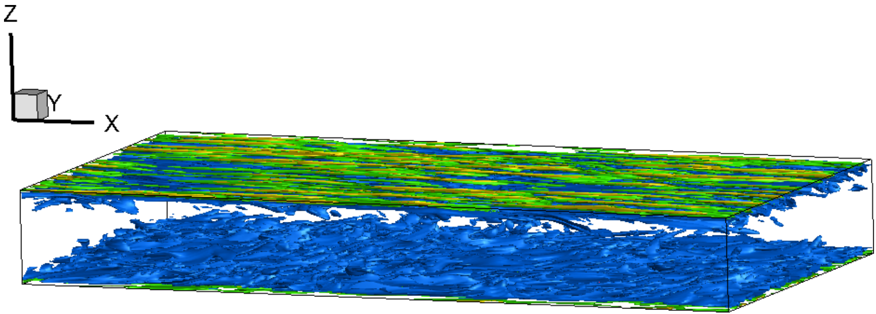

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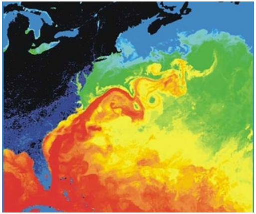
- Numerics require resolution of a vast array of timescales and lengthscales
- Dimentionality is important

How can it be characterized?



Turbulence depends on the dimension – 2D turbulence very different from 3D turbulence. Take 3D turbulence as a paradigm. The fundamental element is the **eddy** – a coherent patch of fluid motion on a particular scale.

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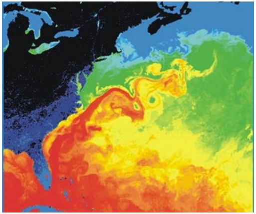

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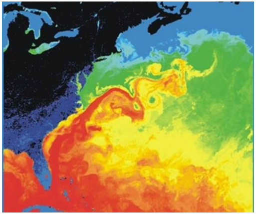

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Best summarized in poetry:

Big whorls have little whorls
Which feed on their velocity,
And little whorls have lesser whorls
And so on to viscosity.

(L.F. Richardson)

How to characterize turbulence

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Crucial information about the spatial structure is then supplied by the following correlation function:

$$R_{ij}(\boldsymbol{x},t) = \langle u_i'(\boldsymbol{x}+\boldsymbol{y},t) u_j'(\boldsymbol{y},t) \rangle,$$

and its Fourier transform $E_{ij}(\boldsymbol{k})$.

Turbulence is generic

Kolmogorov similarity hypotheses:

- On a sufficiently small scale, all turbulence is homogeneous and isotropic
- and can be characterized by universal functional forms...

meaning that

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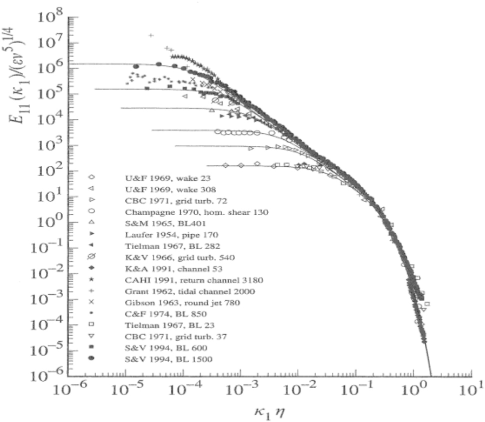

How do we develop an understanding of turbulence?

We aim to answer two key questions:

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To answer these questions, we need a key assumption:

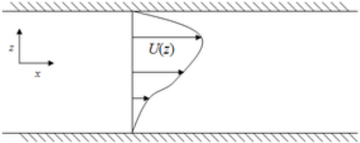
All relevant fluid motions – including turbulence – can be completely characterized by the Navier–Stokes equations:

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = -\nabla p + \mu\nabla^2\boldsymbol{u} + \boldsymbol{f}, \qquad \nabla\cdot\boldsymbol{u} = 0,$$

where $\boldsymbol{f}$ includes the effects of large-scale forcing.

Where does turbulence come from?

We will look at the specific example of **channel flow**.



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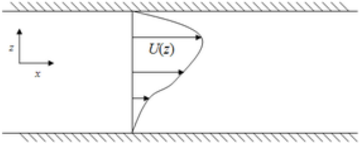

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For unidirectional steady flow with constant forcing (pressure drop) the Navier–Stokes equations have an analytical solution:

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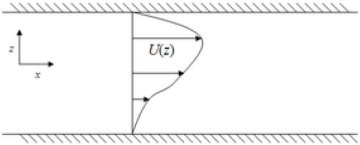

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This suggests a **nondimesionalization** based on the friction velocity $V = \sqrt{(H/2\rho)|dP/dL|}$, with

$$\tilde{u}(\tilde{z}) = Re_*\tilde{z}(1-\tilde{z}),$$

$$Re_* = \frac{\rho V H}{\mu}.$$

Linear stability analysis I

- Introduce a tiny sinusoidal perturbation (wavenumber α) around the base flow.
- Produces pressure and velocity fluctuations that satisfy linearized equations of motion.
- Linearized equations of motion solved via eigenvalue analysis for complex eigenvalue $\lambda = -\mathrm{i}\omega$ (Orr–Sommerfeld equation)

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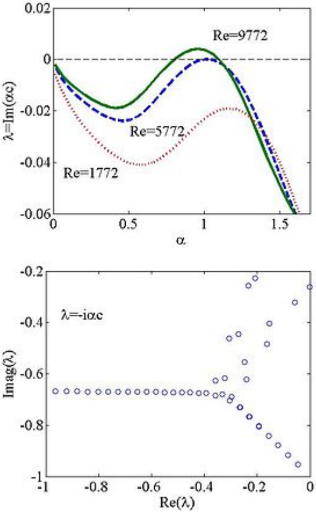

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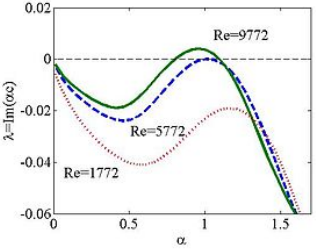

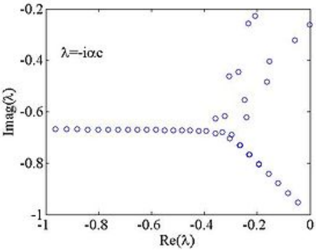

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- Seems like the end of the story, but there is a problem – Transition to turbulence is observed below Re_{*c} – **subcritical transition to turbulence**.
- Subcritical transition in channel flow can be understood using two theoretical tools – **transient growth** and **coherent states**. We can understand these using 2×2 matrices.

Back to ACM 10060

Consider simple two-dimensional autonomous dynamical system:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}).$$

- Fixed points $\mathbf{x_0}$ satisfy $\mathbf{F}(\mathbf{x_0}) = \mathbf{0}$ (base state!)
- Fixed points are classified by their stability: form Jacobian matrix

$$J = \left(\begin{array}{cc} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{array}\right)_{\mathbf{x_0}}.$$

  and compute eigenvalues $\lambda = \mathrm{spec}(J)$.
- If $\Re(\lambda) > 0$ for some eigenvalue, then system is unstable, otherwise it is stable or neutral.

Forcing

- Forcing can be introduced by looking at

$$\dot{\mathbf{x}} = \mu \mathbb{I} \mathbf{x} + \mathbf{F}(\mathbf{x}), \qquad \mu \in \mathbb{R}^{+}.$$

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- Generally, $\Re(\lambda) < 0$ for $\mu < \mu_c$ and $\Re(\lambda) > 0$ for $\mu > \mu_c$ indicating a transition from stability to instability at the critical value μ_c.
- **But this is not the end of the story!**

Transient growth

- Transient growth can occur in linear systems where the Jacobian is **non-normal**:

$$JJ^\dagger - J^\dagger J \neq 0.$$

- There are situations where all eigenvalues are linearly stable but solutions of $\dot{\mathbf{u}} = J\mathbf{u}$ grow rapidly before the eigenvalue theory eventually kicks and and forces

$$\|\mathbf{u}\| \to 0 \text{ as } t \to \infty.$$

- Growth is measured by amplification factor

$$G(t) = \sup_{\substack{\mathbf{u}_0 \\ \|\mathbf{u}_0\|=1}} \|\mathrm{e}^{Jt}\mathbf{u}_0\|$$

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We look at a simple concrete example (motivated by physics) that will make this much less mysterious.

Two-level system – linear theory I

Two-level system:

$$\mathrm{i}\frac{\partial u}{\partial t} = \mathcal{H}u + \mathrm{i}\left(\mu_0\mathbb{I} + \mathcal{G}\right)u, \qquad u \in \mathbb{C}^2,$$

where

$$\mathcal{H} = \begin{pmatrix} E_0 & A \\ A & E_0 \end{pmatrix}, \qquad \mathcal{G} = \mathrm{diag}(-g_1, -g_2).$$

Note that $[\mathcal{H}, \mathcal{G}] \neq 0$ implies that the operator

$$\mathcal{L} = \mathcal{H} + \mathrm{i}\left(\mu_0\mathbb{I} + \mathcal{G}\right)$$

is **non-normal**, with $\left[\mathcal{L}, \mathcal{L}^\dagger\right] \propto g_2 - g_1$.

Two-level system – linear theory II

Eigenvalues: let $u(t) = u_0 \mathrm{e}^{-\mathrm{i}\omega t}$, to give

$$\Omega_{\mathrm{r}} = E_0 \pm \sqrt{4A^2 - (g_1 - g_2)^2}, \qquad \Omega_{\mathrm{i}} = \mu_0 - \tfrac{1}{2}(g_1 + g_2),$$
$$4A^2 > (g_1 - g_2)^2, \qquad \text{Case 1},$$

$$\Omega_{\mathrm{r}} = E_0, \qquad \Omega_{\mathrm{i}} = \mu_0 - \tfrac{1}{2}(g_1 + g_2) \pm \sqrt{(g_1 - g_2)^2 - 4A^2}, \qquad \text{Case 2}.$$

We work in **Case 2** (crossover is called the diabolic point).

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We use

$$\tfrac{1}{2}\frac{d}{dt}\|u\|_2^2 \le \left[\mu_0 - \min(g_1, g_2)\right] \|u\|_2^2$$

to identify subcritical parameter values for the forcing μ_0 where transient growth is possible:

$$\min(g_1, g_2) < \mu_0 < \tfrac{1}{2}(g_1 + g_2) - \sqrt{(g_1 - g_2)^2 - 4A^2}.$$

Introduction of nonlinear terms I

- Transient growth by itself won't induce a subcritical transition because eventually the disturbance will die out.
- The idea is that the transient growth will excite a nonlinear solution:
  - Transient growth excites nonlinear solution,
  - Nonlinear solution has a tendency to decay over time (damping) but this is counteracted by further transient growth.
  - Nonlinear solution is therefore a quasi-steady structure (coherent state)
  - Nonlinear solution can itself be unstable to secondary instability leading to a **cascade** whereby more and more nonlinear solutions of increasing complexity are excited.

We therefore look to add some nonlinear terms to the two-level system to see what might happen...

Introduction of nonlinear terms II

- Nonlinear two-level system:

$$\mathrm{i}\frac{\partial u}{\partial t} = \mathcal{L}u + a \begin{pmatrix} |u_1|^2 & 0 \\ 0 & |u_2|^2 \end{pmatrix} u.$$

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$$\mathrm{i}\frac{\partial u}{\partial t} = \mathcal{L}u + a\begin{pmatrix} |u_1|^2 & 0 \\ 0 & |u_2|^2 \end{pmatrix} u.$$

- We search for a **self-sustained oscillatory solution**:

$$u = R\mathrm{e}^{\mathrm{i}\Omega t}u_0, \qquad \|u_0\|_2^2 = 1, \qquad \Omega \in \mathbb{R}.$$

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- Such a solution can be found for $g_2 < \mu_0 < g_2$: we have $\Omega = E_0 + aR^2$, where R has the special value

$$R^2 = \frac{g_1 - g_2}{a}\sqrt{\frac{1}{X^2} - 1}, \qquad X^2 = -\frac{(\mu_0 - g_1)(\mu_0 - g_2)}{A^2}.$$

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- Floquet analysis reveals that the nonlinear oscillation is always unstable to a secondary instability (exact result).

Numerical solution

Numerical simulation with 8th-order accurate Runge-Kutta scheme, with initial condition $u(t=0) = (A_0/\sqrt{2})(\mathrm{i}, 1)^T$.

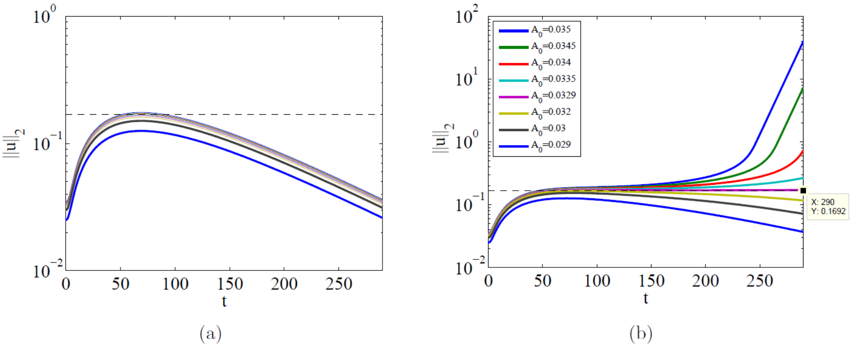
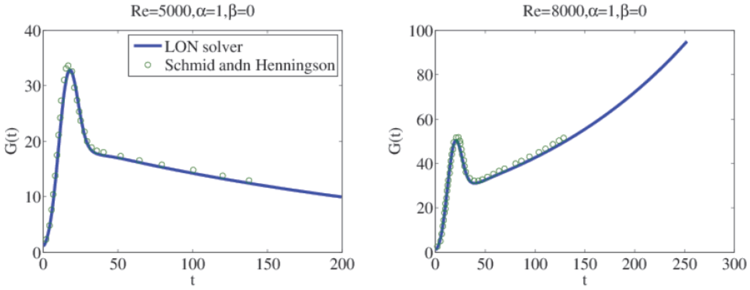


FIG. 2. Solutions of (a) the non-Hermitian linear Schrodinger equation; (b) the non-Hermitian *nonlinear* Schrödinger equation. The initial data are the parameters are the same in (a) and (b).

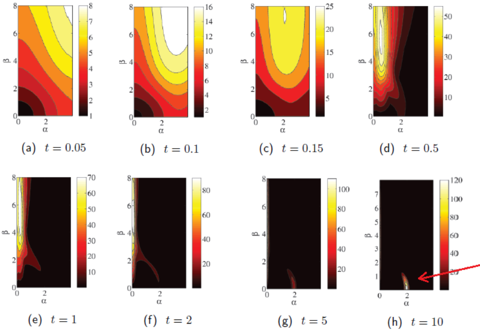
Transient growth is found in Orr–Sommerfeld equation



(a) (b)

Figure 6.2: Validation of our code for the maximum transient growth rate compared to known benchmark case in the literature (data from Reference [SH01]). The small discrepancies between the two datasets are due to errors in scanning and digitizing the data from the reference text.

Transient growth is much more important for 3D modes than for 2D modes



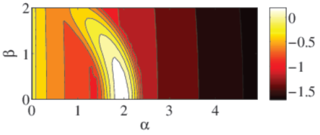


Figure 6.4: Eigenvalue of most-dangerous mode of the Orr–Sommerfeld–Squire equations, with $Re = 5000$. The most-dangerous mode according to eigenvalue analysis (valid as $t \to \infty$) is a streamwise one.

Figure 6.3: Time evolution of the optimal transient growth rate as a function of the wavenumbers α (streamwise) and β (spanwise). Between $t = 0.1$ and $t = 10$ the optimal disturbance moves from being spanwise-dominated to streamwise-dominated.

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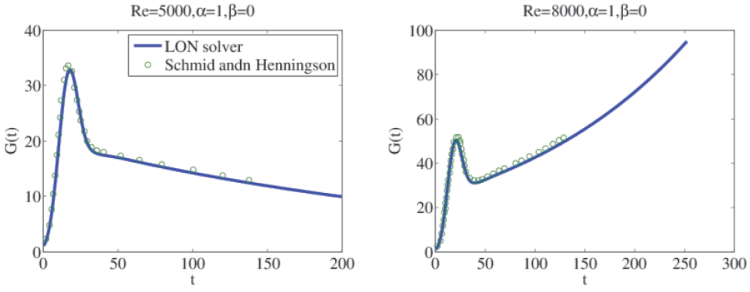

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Coherent structures are found in channel flows as unstable travelling waves

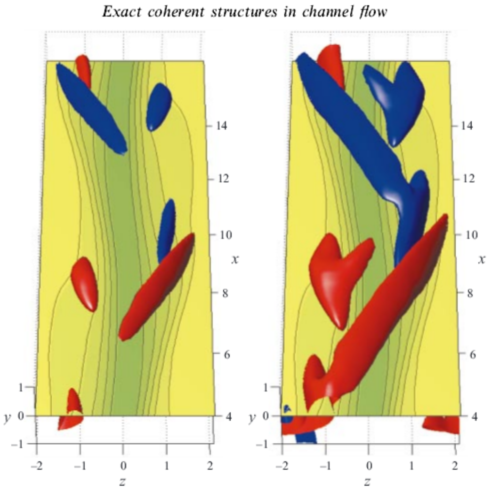

FIGURE 5. Lower branch at $Re = 415$ ($R_\tau \approx 58$). Level curves of streamwise velocity u at $y = 0$ overlayed with isosurfaces of streamwise vorticity (Left: $\pm 60\%\max[\omega_x(x,y,z)]$), right: $\pm 40\%\max[\omega_x(x,y,z)]$).

[Waleffe, JFM, 2001]

The second question

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Answer is yes, but **direct numerical simulation** is extremely costly:

- All energy-containing scales down to the Kolmogorov microscale η need to be resolved.
- Back-of-the envelope scaling calculations indicate that

$$\eta \sim Re^{-3/4}$$

  where Re is the Reynolds number based on the large-scale forcing and domain size.
- Hence, $\Delta x \sim Re^{-3/4}$ in a numerical simulation, requiring N_T gridpoints, where

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$$N_T \sim \Delta x^{-3} \sim Re^{9/4}.$$
- For flow around an aeroplane, we might have $Re = 2 \times 10^7$, leading to a requirement of
$$N_T \sim 10^{17} \text{ gridpoints.}$$

Turbulence modelling

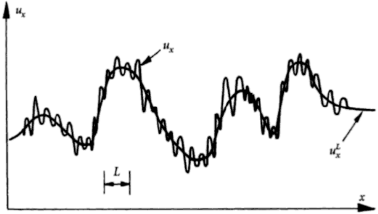
- DNS are restricted to low-to-intermediate Reynolds numbers.
- Also, for doing parameter studies, running hundreds of DNS may be infeasible – even at the low-Reynolds number end.
- Requirement for turbulence models more sophisticated than two-level systems (!) but less computationally intensive than high-res DNS.
- The best tradeoff so far (used commonly by engineers, scientists, and designers) is **large-eddy simulation**.

Large-eddy simulation (LES)

Filter the velocity field:

$$\implies \boldsymbol{u}(\boldsymbol{x},t) = \overline{\boldsymbol{u}}(\boldsymbol{x},t) + \boldsymbol{u}'(\boldsymbol{x},t)$$

$$\overline{\boldsymbol{u}}(\boldsymbol{x},t) = \int_{\Omega} G(\boldsymbol{r})\, \boldsymbol{u}(\boldsymbol{x}-\boldsymbol{r},t)\, d\boldsymbol{r}$$



Apply the filtering process to the Navier-Stokes equations:

$$\frac{\partial \overline{u_i}}{\partial t} + \overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\mu\left(\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i}\right)\right) - \frac{\partial \overline{\tau_{ij}}}{\partial x_j}, \qquad \frac{\partial \overline{u_i}}{\partial x_i} = 0$$

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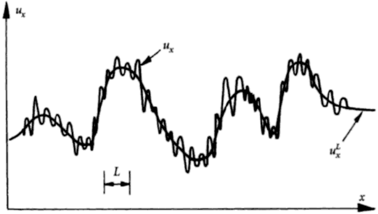

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Closure problem – additional stresses $\overline{\tau_{ij}}$ to be modelled

Closure problem – Smagorinsky model

Smagorinsky (1963) proposed:

$$\overline{\tau_{ij}} = -\nu_t \left(\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i} \right) = -2\nu_t \overline{\boldsymbol{s}_{ij}}$$

$$\implies \frac{\partial \overline{u_i}}{\partial t} + \overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \frac{1}{\rho}\frac{\partial}{\partial x_j}\left((\mu + \nu_t) \left(\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i} \right) \right)$$

$$\to \nu_t = \left(C_S \Delta \phi_w (z) \right)^2 |\overline{\boldsymbol{s}}|, \quad |\overline{\boldsymbol{s}}| = \sqrt{2 \left(\overline{\boldsymbol{s}_{ij}} \right) \left(\overline{\boldsymbol{s}_{ij}} \right)}$$

$\implies$ Implement eddy viscosity, initialise system and run the simulation!

Computational framework

Even a large eddy-simulation can be challenging to implement numerically – In-house implementation uses TPLS computational framework developed by yours truly:

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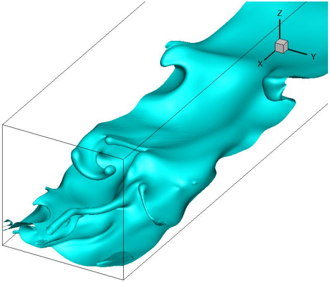

HECToR: UK National Supercomputing Service

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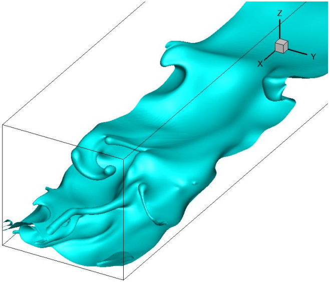

> Single-phase version available for demonstration purposes, and incorporates the Smagorinsky-LES model.

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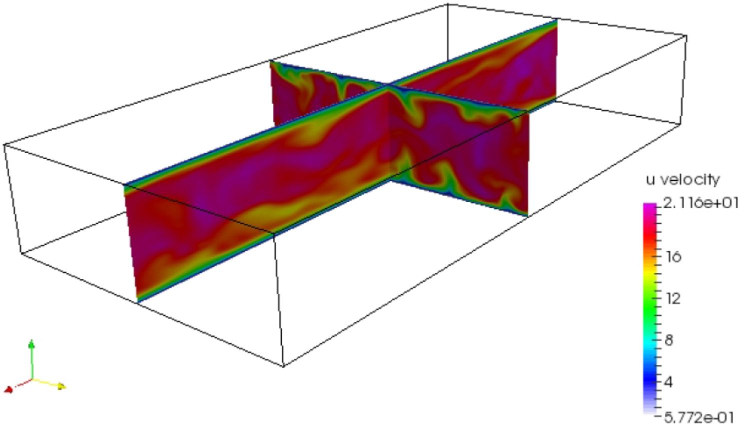
archer

ICHEC
Irish Centre for High-End Computing

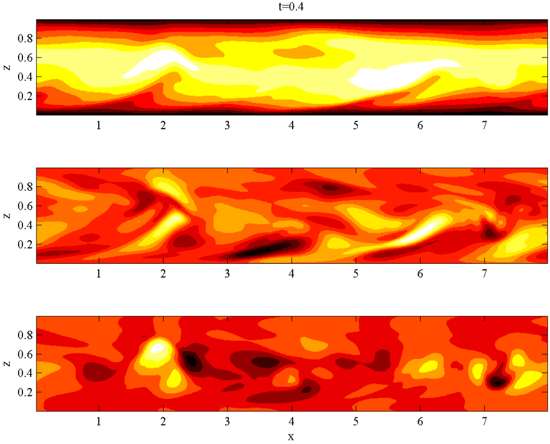
Aside – why write your own code?

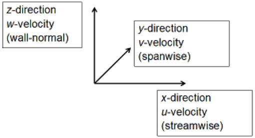
https://www.youtube.com/watch?v=gzSMkKef9nQ

LES Results – instantaneous snapshots I



LES Results – instantaneous snapshots II

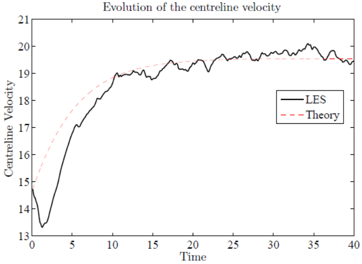




Top – streamwise
Middle – spanwise
Bottom – wall-normal
All taken in a particular xz plane

LES Results – turbulent statistics I

- Spacetime average of physical quantities is steady – forcing and dissipation in balance on average.



Prediction (S-E): $U_C \approx 19.56$
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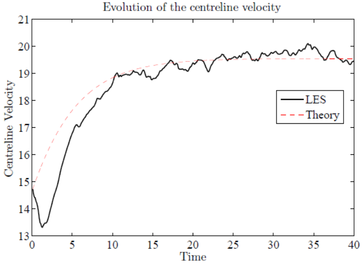

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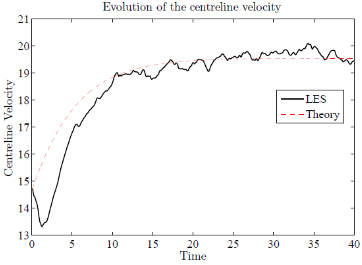

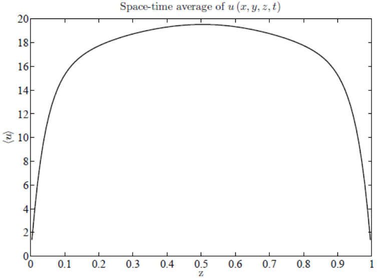

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	U_{mean}	U_C/U_{mean}
Kim et al. (1986)	15.63	1.16
LES	16.75	1.167

LES Results – law of the wall

- Reynolds averaging also gives a balanace equation for the mean velocity:

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\tau_R + \mu\frac{\mathrm{d}U_0}{\mathrm{d}z}\right) - \frac{\mathrm{d}P}{\mathrm{d}L} = 0.$$

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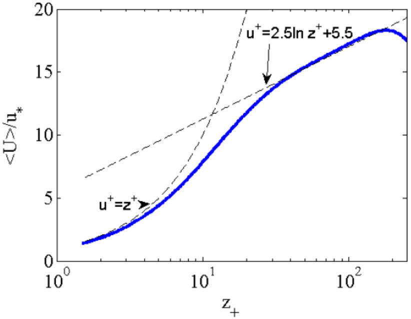

Conclusions

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- Can be understood by combining transient growth with coherent structures
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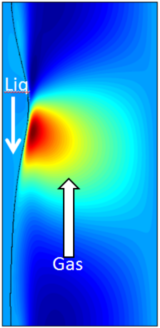
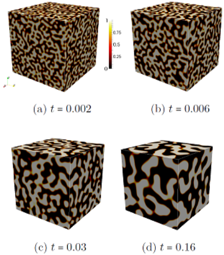
- But TPLS is much more than this...

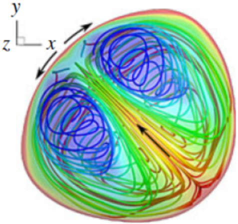
TPLS has been used in a wide variety of applications



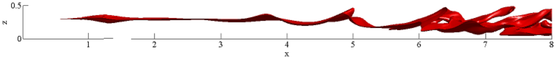
Countercurrent gas-liquid flows



Phase separation in binary liquids



Evaporation of non-spherical droplets



Waves in two-layer flows

...and is available as open-source software:

`http://sourceforge.net/projects/tpls/`