Chapter 14

The 1-D wave equation: Causality

Overview

In this section we show, by examples, that information is propagated at speed c in the wave equation. Here, 'information' means initial data; we will see shortly what is meant by propagation.

14.1 Example I

Consider the wave equation

$$u_{tt} = u_{xx}$$

with initial data

$$\begin{aligned} u(x, t=0) &= f(x) = \begin{cases} F(x), & |x| \le 1, \\ 0, & |x| > 1 \end{cases}, \\ u_t(x, t=0) &= g(x) = 0. \end{aligned}$$

and with wave speed $c = 1$. We are to solve for the wave. By d'Alembert's formula, the solution is

$$u(x,t) = \tfrac{1}{2}\left[f(x+t) + f(x-t)\right].$$

We need to identify where $|x+t|$ and $|x-t|$ are less than one; outside of these regions the solution is zero.

- Case 1: $|x+t| \le 1$ AND $|x-t| \le 1$. Along the lines where the **equalities** hold, $|x+t| = 1$

and $|x-t|=1$. These lines represent the boundaries of the region of interest:

$$-1 \leq x-t \leq 1, \qquad -1 \leq x+t \leq 1.$$

Note also that $dx/dt = \pm 1 = \pm c$ along these lines, i.e. they are **characteristic lines** that give a trajectory moving at the wave speed.

We pick out the pertinent boundary lines:

$$t \leq x+1, \qquad t \leq 1-x.$$

These are lines with slopes ± 1 and a $y-axis$ intercept at 1. The region R_1 is below these lines, and above the x-axis ($t=0$).

- Case 2: $|x-t| \leq 1$ only. In other words, $-1 \leq x-t \leq 1$. The boundaries of this region are

$$t \leq x+1,$$

$$t \geq x-1.$$

These are characteristics.

- Case 3: $|x+t| \leq 1$ only. In other words, $-1 \leq x+t \leq 1$. The boundaries of this region are

$$t \leq 1-x$$

$$t \geq -1-x.$$

Next, we plot these different regions in spacetime (Fig. 14.1).

- In region R_1, $|x+t| \leq 1$ AND $|x-t| \leq 1$;
- In region R_2, $|x-t| \leq 1$ only;
- In region R_3, $|x+t| \leq 1$ only;
- Outside of these regions, $|x+t|$ AND $|x-t|$ both exceed 1 (> 1).

Thus,

$$u(x,t) = \begin{cases} \frac{1}{2}\left[F(x+t)+F(x-t)\right], & (x,t) \in R_1, \\ \frac{1}{2}F(x-t), & (x,t) \in R_2, \\ \frac{1}{2}F(x+t), & (x,t) \in R_3, \\ 0, & \text{otherwise.} \end{cases}$$

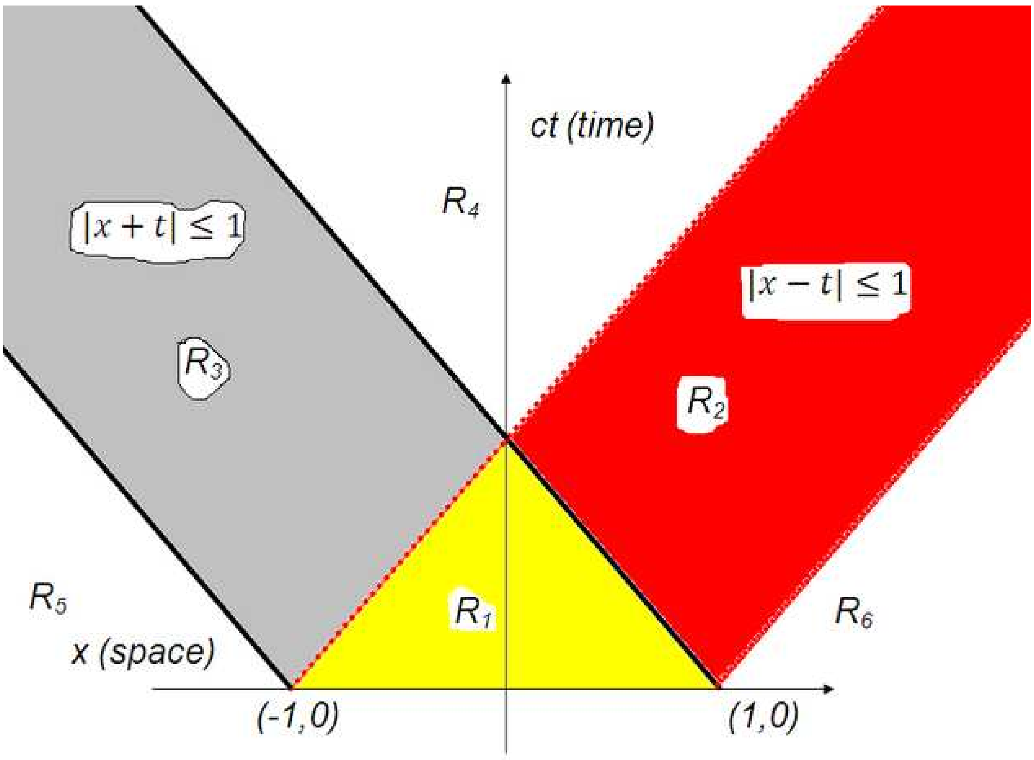


Figure 14.1: The different regions where $|x \pm t| \leq 1$.

Physical interpretation

- The initial, compactly-supported disturbance remains compactly supported for all time. The support never exceeds $x = 1 + ct$ and $x = -1 - ct$, which are characteristics $dx/dt = \pm c$.
- In other words, the equations $x = x_{0R} + ct = 1 + t$ and $x = x_{0L} - ct = -1 - t$ are an envelope within which information is carried forwards in time.
- Outside of this envelope, no information is carried forwards.
- This is the notion of **causality:** The initial solution affects the solution at a later time, within the boundaries set by the characteristics $x = x_{0R} + ct$ and $x = x_{0L} - ct$.
- In other words, causality demands that a compactly-supported initial condition always remain compactly supported, and that support should depend on the initial conditions and the characteristics.
- As can be seen from Fig. 14.1, $F(x - ct)$ represents a right-travelling disturbance, because the domain $|x - ct| \leq 1$ extends into the right half of the spacetime plane.

14.2 Example II

Consider the wave equation

$$u_{tt} = u_{xx}$$

with initial data

$$\begin{aligned} u(x, t = 0) &= f(x) = 0, \\ u_t(x, t = 0) &= g(x) = \begin{cases} G(x) := \cos^2(\pi x/2), & |x| \le 1, \\ 0, & |x| > 1 \end{cases}, \end{aligned}$$

and with wave speed $c = 1$. We are to solve for the wave. By d'Alembert's formula, the solution is

$$u(x, t) = \tfrac{1}{2}\left[f(x + t) + f(x - t)\right] + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,\mathrm{d}s.$$

We need to identify where $|x + t|$ and $|x - t|$ are less than one; outside of these regions the solution is zero. But we have already done this:

1. In region R_1, $|x + t| \le 1$ AND $|x - ct| \le 1$;

2. In region R_2, $|x - t| \le 1$ only;

3. In region R_3, $|x + t| \le 1$ only;

4. In region R_4, $x - t \le -1$ and $x + t \ge 1$;

5. In region R_5, $x + t \le -1$;

6. In region R_6, $x - t \ge 1$.

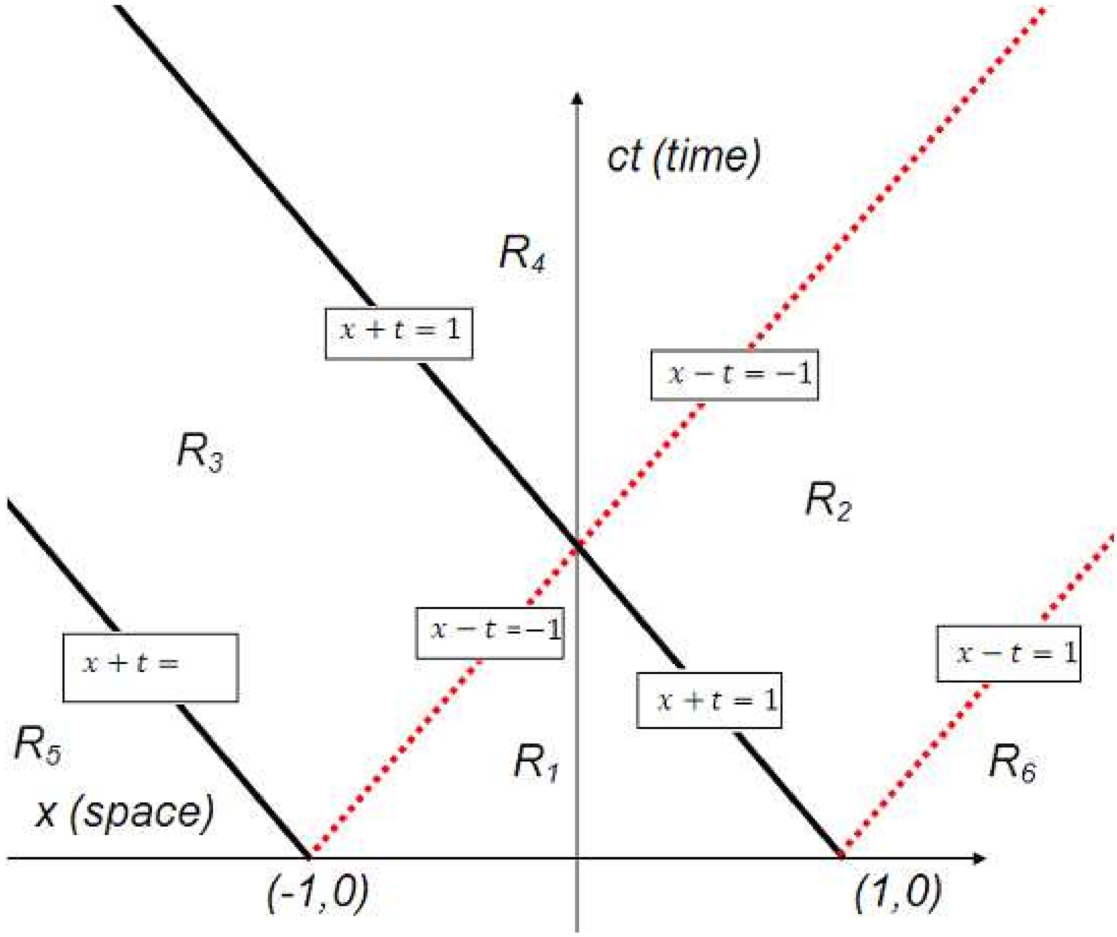
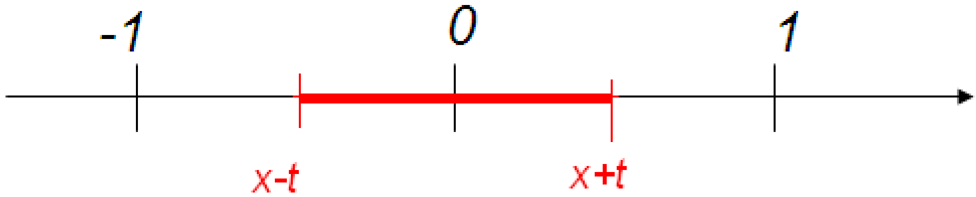


Figure 14.2: The different regions where $|x \pm t| \leq 1$.



Region 1: $|x+t| \le 1$ and $|x-t| \le 1$. This implies that

$$-1 \le x-t \le x+t \le 1.$$

Do the G-integral. Note:

$$\int_a^b \cos^2(\pi x/2) \mathrm{d}x = \int_a^b \tfrac{1}{2}\left[1+\cos(\pi x)\right] \mathrm{d}x = \tfrac{1}{2}(b-a) + \tfrac{1}{2}\frac{\sin(\pi b)-\sin(\pi a)}{\pi}.$$

Inside region 1,

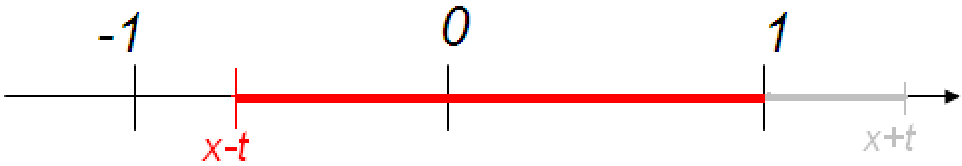
$$-1 \le x-t \le x+t \le 1,$$

so $G(s) = g(s)$ everywhere in the integral:

$$\begin{aligned}\int_{x-t}^{x+t} G(s)\,\mathrm{d}s &= \int_{x-t}^{x+t} \cos^2(\pi s/2)\,\mathrm{d}s, \\ &= t + \frac{\sin\left(\pi(x+t)\right) - \sin\left(\pi(x-t)\right)}{2\pi}, \\ &= t + \frac{1}{\pi}\cos(\pi x)\sin(\pi t).\end{aligned}$$

Finally, in region 1,

$$u_1(x,t) = 0 + \tfrac{1}{2}t + \frac{1}{2\pi}\cos(\pi x)\sin(\pi t).$$



Region 2: $|x-t| \leq 1$. Inspection of Fig. 14.2 shows that the region boundaries are

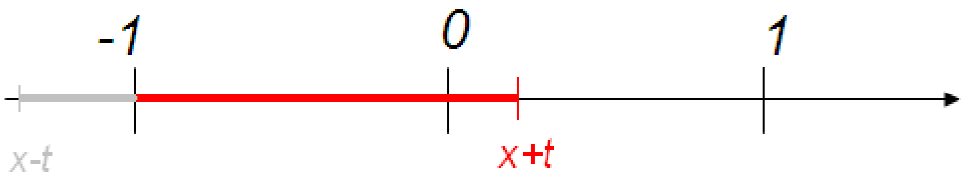
$$-1 \leq x-t \leq 1 \text{ AND } x+t > 1,$$

so that $x-t \geq -1$ and $x+t \geq 1$. Now $G(s) = g(s)$ for $s \in [x-t, 1]$ and is zero elsewhere. Hence,

$$\begin{aligned}
\int_{x-t}^{x+t} G(s)\,\mathrm{d}s &= \int_{x-t}^{1} \cos^2(\pi s/2)\,\mathrm{d}s, \\
&= \tfrac{1}{2}\left[1-(x-t)\right] + \frac{\sin(\pi) - \sin(\pi(x-t))}{2\pi}, \\
&= \tfrac{1}{2}\left[1-x+t\right] - \frac{\sin(\pi(x-t))}{2\pi}.
\end{aligned}$$

Finally, in region 2,

$$u_2(x,t) = \tfrac{1}{4}\left[1-x+t\right] - \frac{\sin(\pi(x-t))}{4\pi}.$$



Region 3: $|x+t| \leq 1$. Inspection of Fig. 14.2 shows that the region boundaries are

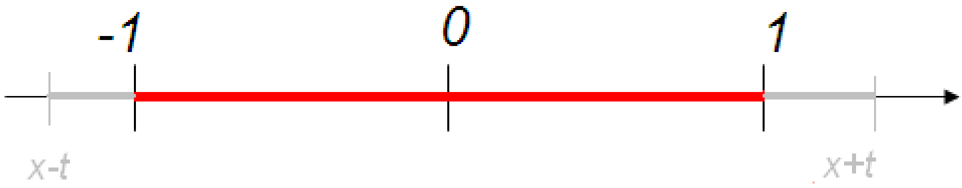
$$-1 \leq x+t \leq 1 \text{ AND } x-t < -1,$$

so that $x-t \leq -1$ and $x+t \leq 1$. Now $G(s) = g(s)$ for $s \in [-1, x+t]$ and is zero elsewhere. Hence,

$$\begin{aligned} \int_{x-t}^{x+t} G(s)\,\mathrm{d}s &= \int_{-1}^{x+t} \cos^2(\pi s/2)\,\mathrm{d}s, \\ &= \tfrac{1}{2}\left[x+t-(-1)\right] + \frac{\sin\left(\pi(x+t)\right) - \sin\left(\pi(-1)\right)}{2\pi}, \\ &= \tfrac{1}{2}\left[x+t+1\right] + \frac{\sin\left(\pi(x+t)\right)}{2\pi}. \end{aligned}$$

Finally, in region 3,

$$u_2(x,t) = \tfrac{1}{4}\left[x+t+1\right] + \frac{\sin\left(\pi(x+t)\right)}{4\pi}.$$



Region 4: Fig. 14.2,

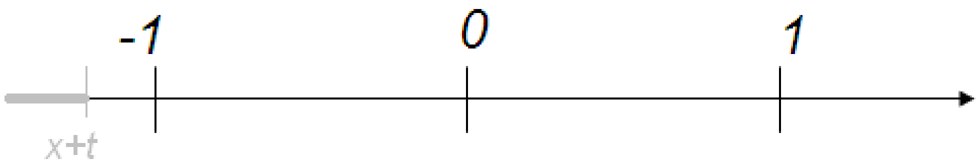
$$x - t \leq -1, \qquad x + t \geq 1$$

so $G(s) = g(s)$ for $s \in [-1, 1]$ and is zero elsewhere.

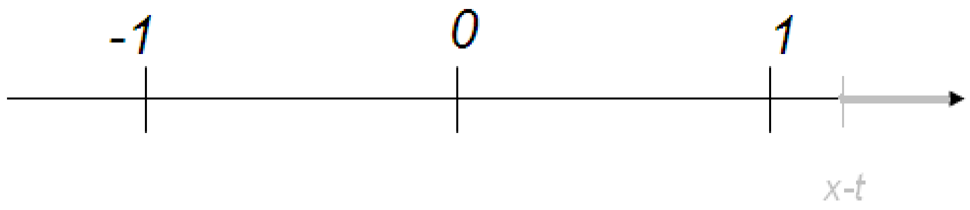
$$\begin{aligned}\int_{x-t}^{x+t} G(s)\,\mathrm{d}s &= \int_{-1}^{1} \cos^2(\pi s/2)\,\mathrm{d}s, \\ &= \tfrac{1}{2}\left[1-(-1)\right] + \frac{\sin(\pi) - \sin(\pi(-1))}{2\pi}, \\ &= 1\end{aligned}$$

Finally, in region 4,

$$u_4(x,t) = \tfrac{1}{2}.$$



Region 5: $x + t < -1$, hence $G(s) = 0$.



Region 6: $x - t > 1$, hence $G(s) = 0$.

Putting it all together,

$$u(x,t)=\begin{cases}u_1(x,t), & (x,t)\in R_1,\\ u_2(x,t), & (x,t)\in R_2,\\ u_3(x,t), & (x,t)\in R_3,\\ u_4(x,t), & (x,t)\in R_4,\\ u_5(x,t), & (x,t)\in R_5,\\ u_6(x,t), & (x,t)\in R_6,\end{cases}$$

or,

$$u(x,t)=\begin{cases}\frac{1}{2}t+\frac{1}{2\pi}\cos(\pi x)\sin(\pi t), & (x,t)\in R_1,\\ \frac{1}{4}\left[1-x+t\right]-\frac{\sin(\pi(x-t))}{4\pi}, & (x,t)\in R_2,\\ \frac{1}{4}\left[x+t+1\right]+\frac{\sin(\pi(x+t))}{4\pi}, & (x,t)\in R_3,\\ \frac{1}{2}, & (x,t)\in R_4,\\ 0, & (x,t)\in R_5,\\ 0, & (x,t)\in R_6,\end{cases}$$

Notes:

- Region 4 gives a contribution here. If there is no initial velocity ($u_t(x,t=0)=0$), there is no contribution from this region.
- I have sketched the d'Alembert solution in Fig. 14.3, using the code wavesolve__exact.m.
- There is also available my webpage, a finite-difference code integrate__sde.m. One can test the finite-difference code and the exact-solution code and they give the same answer, for all times.