Chapter 11

The 1-D wave equation

Overview

The wave equation describes linear oscillations in a generic field $u(x,t)$:

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},$$

where c is the propagation speed of the oscillations. Topics: derivation; solution through separation of variables; energy conservation.

11.1 Derivation

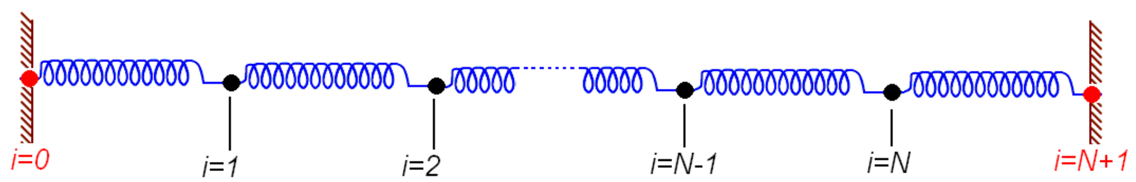

Figure 11.1: N particles connected in a line via identical springs (clamped boundary conditions).

Consider N identical particles arrayed in a line, and connected together by identical springs (Fig. 11.1). The equilibrium position of the i^{th} particle is $x_i = i\Delta x$, with $i = 1, 2, \cdots N$, and the departure from equilibrium is small and equal to y_i. The potential energy of such a system is

$$\mathcal{U}(y_1, \cdots, y_N) = \tfrac{1}{2}k \sum_{i=1}^{N-1} (y_{i+1} - y_i)^2 + \text{Boundary terms.}$$

The boundary terms can be taken care of by forcing the displacements $y_0 = y_{N+1} = 0$. Thus,

$$\mathcal{U}(y_1, \cdots, y_N) = \tfrac{1}{2}k\sum_{i=0}^{N}\left(y_{i+1}-y_i\right)^2, \qquad y_0 = y_{N+1} = 0.$$

Hence, at interior points ($i = 1, 2, \cdots, N$), Newton's law gives

$$\begin{aligned} m\frac{d^2 y_i}{dt^2} &= -\frac{\partial\mathcal{U}}{\partial y_i} = -k\left[-(y_{i+1}-y_i) + (y_i - y_{i-1})\right], \\ &= k(y_{i+1}-y_i) - k(y_i - y_{i-1}). \end{aligned}$$

The mass of each oscillator is $m = \rho\Delta x$, where ρ is the constant (linear density) of the system. Thus,

$$\frac{d^2 y_i}{dt^2} = \frac{k\Delta x}{\rho}\frac{(y_{i+1}-y_i)-(y_i-y_{i-1})}{(\Delta x)^2}.$$

We identify $T = k\Delta x$ as the tension in the system of springs. Thus,

$$\frac{d^2 y_i}{dt^2} = \frac{T}{\rho}\frac{(y_{i+1}-y_i)-(y_i-y_{i-1})}{(\Delta x)^2}.$$

Taking $\Delta x \to 0$ ($N \to \infty$) gives

$$\frac{\partial^2 y(x,t)}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 y(x,t)}{\partial x^2},$$

where we identify

$$c^2 = \frac{T}{\rho}.$$

Now

$$[c]^2 = \frac{\text{Force}}{\text{Mass/Length}} = \frac{\text{Mass Length/Time}^2}{\text{Mass/Length}} = \frac{\text{Length}^2}{\text{Time}^2},$$

and c is clearly a velocity: it is the velocity at which a wave of small oscillations propagates along the spring system. A similar treatment of other systems yields the same linear wave equation. For example, for small oscillations in a gas, the linear wave equation is satisfied, with

$$c_{\text{gas}}^2 = \frac{\gamma p_0}{\rho},$$

where

- γ is the (nondimensional) ratio of specific heats;
- p_0 is the equilibrium pressure;
- ρ is the mass per unit volume.

In any case, the generic equation we study in this section is

$$\frac{1}{c^2}\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2}.$$

11.2 Boundary and initial conditions

The most common kind of boundary conditions is the requirement that the oscillations at the end points of the domain $\Omega = [0, L]$ should be zero:

$$u(x = 0, t > 0) = u(x = L, t > 0) = 0.$$

In the language of Ch. 8, these are the **homogeneous Dirichlet conditions**. We need **two** boundary conditions because the equation is second-order in space. For the diffusion equation, we needed only one initial condition, because the equation was first-order in time. However, the wave equation is second-order in time, so we need **two initial conditions**, usually taken to be

$$\begin{aligned} u(x, t = 0) &= f(x), \qquad 0 < x < L, \\ u_t(x, t = 0) &= g(x), \qquad 0 < x < L. \end{aligned}$$

11.3 Separation of variables

Consider a taut string, such as a violin string, that is plucked according to the initial conditions

$$\begin{aligned} u(x, t = 0) &= f(x), \qquad 0 < x < L, \\ u_t(x, t = 0) &= g(x), \qquad 0 < x < L. \end{aligned}$$

The string is fixed at the end points, $u(0) = u(L) = 0$. Solve for the vibrations in the string.

We solve by separation of variables:

$$u(x, t) = X(x)T(t).$$

Substitution into the wave equation gives

$$\frac{1}{c^2}T''(t)X(x) = X''(x)T(t).$$

Dividing by $X(x)T(t)$ gives

$$\frac{1}{c^2}\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)}.$$

Since the LHS is a function of time alone and the RHS is a function of space alone, the only way for this equation to be satisfied is if both sides are in fact equal to a constant:

$$\frac{1}{c^2}\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda.$$

Let us also substitute the trial solution into the BCs and the ICs:

$$\begin{array}{rl} \text{Initial condition:} & u(x,t=0) = X(x)T(0) = f(x), \qquad 0 < x < L, \\ \text{Initial condition:} & u_t(x,t=0) = X(x)T'(0) = g(x), \qquad 0 < x < L, \\ \text{Boundary condition:} & T(t)X(0) = T(t)X(L) = 0 \end{array}$$

Solving for $X(x)$

Focussing on the $X(x)$-equations, we have:

$$\begin{aligned} \frac{1}{X}\frac{d^2X}{dx^2} &= -\lambda, \qquad 0 < x < L, \\ X(0) &= X(L) = 0. \end{aligned}$$

Equation in the **bulk** $0 < x < L$:

$$\frac{d^2X}{dx^2} + \lambda X = 0, \qquad (*)$$

Different possibilities for λ:

1. $\lambda = 0$. Then, the solution is $X(x) = Ax + B$, with $dX/dx = A$. However, the BCs specify $X(0) = 0$, hence $B = 0$. They also specify $X(L) = 0$, hence $A = 0$. Thus, only the trivial solution remains, in which we have no interest.

2. $\lambda < 0$. Then, the solution is $X(x) = Ae^{\mu x} + Be^{-\mu x}$, where $\mu = \sqrt{-\lambda}$. The BCs give

   $$A + B = Ae^{\mu L} + Be^{-\mu L} = 0.$$

   Grouping the first two of these equations together gives

   $$A = -B\frac{1 - e^{-\mu L}}{1 - e^{\mu L}}.$$

But $A+B=0$, hence

$$\begin{aligned} B\left[1-\frac{1-e^{-\mu L}}{1-e^{\mu L}}\right] &= 0, \\ B\left[\frac{1-e^{\mu L}-(1-e^{-\mu L})}{1-e^{\mu L}}\right] &= 0, \\ B\left[-e^{\mu L}+e^{-\mu L}\right] &= 0, \\ B\sinh(\mu L) &= 0, \end{aligned}$$

which has only the trivial solution.

3. Thus, we are forced into the third option: $\lambda > 0$.

Solving Eq. (*) with $\lambda > 0$ gives

$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x),$$

with boundary condition

$$A\cdot 1 + B\cdot 0 = A\cos(\sqrt{\lambda}L) + B\sin(\sqrt{\lambda}L) = 0.$$

Hence, $A=0$. Grouping the second and third equations in this string together therefore gives

$$B\sin(\sqrt{\lambda}L) = 0.$$

Of course, $B=0$ is a solution, but this is the trivial one. Therefore, we must try to solve

$$\sin(\sqrt{\lambda}L) = 0.$$

This is possible, provided

$$\sqrt{\lambda}L = n\pi, \qquad n\in\{1,2,\cdots\}.$$

Thus,

$$\lambda = \lambda_n = \frac{n^2\pi^2}{L^2},$$

and

$$X(x) = B\sin\left(\frac{n\pi x}{L}\right),$$

where B is a constant of integration.

Solving for $T(t)$

Now substitute $\lambda_n = n^2\pi^2/L^2$ back into the $T(t)$-equation:

$$\frac{1}{T}\frac{dT^2}{dt^2} = -\lambda c^2 = -\lambda_n c^2.$$

Solving give

$$T(t) = C\cos(c\sqrt{\lambda_n}t) + D\sin(c\sqrt{\lambda_n}t).$$

Putting it all together

Recall the ansatz:

$$u(x,t) = X(x)T(t).$$

Thus, we have a solution

$$X(x)T(t) = B\sin\left(\frac{n\pi x}{L}\right)\left[C\cos(c\sqrt{\lambda_n}t) + D\sin(c\sqrt{\lambda_n}t)\right].$$

Re-labelling the constants, this is

$$X_n(x)T_n(t) = \sin\left(\frac{n\pi x}{L}\right)\left[A_n\cos(c\sqrt{\lambda_n}t) + B_n\sin(c\sqrt{\lambda_n}t)\right].$$

The label n is just a label on the solution. However, each $n = 1, 2, \cdots$ produces a different solution, linearly independent of all the others. We can add all of these solutions together to obtain a **general solution** of the PDE:

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} X_n(x)T_n(t), \\ &= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[A_n\cos(c\sqrt{\lambda_n}t) + B_n\sin(c\sqrt{\lambda_n}t)\right]. \end{aligned}$$

We are almost there. However, we still need to take care of the initial conditions. First IC:

$$\begin{aligned} u(x,t=0) &= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[A_n\cos(c\sqrt{\lambda_n}t) + B_n\sin(c\sqrt{\lambda_n}t)\right]_{t=0}, \\ &= \sum_{n=1}^{\infty} A_n\sin\left(\frac{n\pi x}{L}\right), \\ &= f(x). \end{aligned}$$

But the functions

$$\left\{ \sin\left(\frac{n\pi x}{L}\right) \right\}_{n=1}^{\infty}$$

are orthogonal on $[0, L]$:

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) \mathrm{d}x = \frac{L}{2}\delta_{mn}.$$

Thus, multiply both sides by $\sin(m\pi x/L)$ and integrate:

$$\begin{aligned}
\int_0^\pi f(x) \sin\left(\frac{m\pi x}{L}\right) \mathrm{d}x &= \int_0^\pi \sum_{n=1}^{\infty} A_n \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right), \\
&= \sum_{n=1}^{\infty} A_n \int_0^\pi \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right), \\
&= \sum_{n=1}^{\infty} A_n \frac{L}{2}\delta_{m,n}, \\
&= \frac{A_n L}{2}.
\end{aligned}$$

Hence,

$$A_n = \frac{2}{L}\int_0^\pi f(x) \sin\left(\frac{m\pi x}{L}\right) \mathrm{d}x.$$

Second IC:

$$\begin{aligned}
u_t(x, t=0) &= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \frac{d}{dt}\left\{ \left[A_n \cos(c\sqrt{\lambda_n}t) + B_n \sin(c\sqrt{\lambda_n}t)\right] \right\}_{t=0}, \\
&= \sum_{n=1}^{\infty} B_n c\sqrt{\lambda_n} \sin\left(\frac{n\pi x}{L}\right), \\
&= \sum_{n=1}^{\infty} \frac{B_n n\pi c}{L} \sin\left(\frac{n\pi x}{L}\right), \\
&= g(x).
\end{aligned}$$

Taking the scalar product with $\sin(m\pi x/L)$, we get

$$\int_0^L g(x) \sin\left(\frac{m\pi x}{L}\right) \mathrm{d}x = \frac{L}{2}\frac{B_n n\pi c}{L},$$

hence

$$B_n = \frac{L}{n\pi c}\frac{2}{L}\int_0^L g(x) \sin\left(\frac{m\pi x}{L}\right) \mathrm{d}x.$$

and, substituting back into the general solution, we have

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[A_n \cos(c\sqrt{\lambda_n}t) + B_n \sin(c\sqrt{\lambda_n}t)\right], \qquad (11.1)\\ A_n &= \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x)\,\mathrm{d}x,\\ B_n &= \frac{L}{n\pi c}\frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) g(x)\,\mathrm{d}x. \end{aligned}$$

which is a solution to the wave equation that satisfies the boundary and initial conditions.

Note: Proving that this series converges to the solution is difficult, because we do not have decaying exponentials like $e^{-n^2\pi^2 Dt/L^2}$ as in the diffusion equation, thus making it difficult to apply the Weirstrass M-test

11.4 Physical interpretation of solution

We have found the following solution to the wave equation:

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[A_n \cos\left(n\frac{c\pi}{L}t\right) + B_n \sin\left(n\frac{c\pi}{L}t\right)\right],$$

which vanishes at the boundaries $u(0) = u(L) = 0$.

- The component
$$\sin\left(\frac{n\pi x}{L}\right)\left[A_n \cos\left(n\frac{c\pi}{L}t\right) + B_n \sin\left(n\frac{c\pi}{L}t\right)\right]$$
is called the n^{th} **normal mode of vibration**.

- The solution is a sum over all normal modes.

- Each normal mode is a periodic function of time, with period
$$n\frac{c\pi}{L}\tau_n = 2\pi \implies \tau_n = \frac{2}{n}\frac{L}{c},$$

- The frequency of a normal mode is given by

$$
\begin{aligned}
\omega_n &= \frac{2\pi}{\tau_n}, \\
&= 2\pi \frac{nc}{2L}, \\
&= 2\pi \frac{n}{2L}\sqrt{\frac{T}{\rho}},
\end{aligned}
$$

upon restoration of the original interpretation of the wave speed. This is probably the nicest result of high-school physics: **Modes of vibration of a string are periodic, and each frequency is an integer multiple of a basic or fundamental frequency, given by**

$$
\omega_1 = 2\pi \frac{1}{2L}\sqrt{\frac{T}{\rho}},
$$

- In a given complex disturbance (i.e. multiple frequencies), each mode is characterised by its frequency ω_n and by the quantities A_n and B_n, which tell us the intensity of the contribution made by the n^{th} normal mode. However, we can re-write the disturbance:

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[A_n \cos\left(n\frac{c\pi}{L}t\right) + B_n \sin\left(n\frac{c\pi}{L}t\right)\right], \\
&= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) C_n \sin\left(n\frac{c\pi}{L}t + \gamma_n\right).
\end{aligned}
$$

The quantity

$$
C_n^2 = A_n^2 + B_n^2
$$

is thus the **amplitude** of the n^{th} normal mode and

$$
\gamma_n = \arctan(A_n/B_n)
$$

is its **phase**.

Note: Let

$$
z = B_n + \mathrm{i}A_n,
$$

and write $\theta_n = n\pi ct/L$, and

$$
\cos\theta_n + \mathrm{i}\sin\theta_n = \mathrm{e}^{\mathrm{i}\theta_n}.
$$

We have

$$\begin{aligned} z\mathrm{e}^{\mathrm{i}\theta_n} &= (B_n + \mathrm{i}A_n)(\cos\theta_n + \mathrm{i}\sin\theta_n), \\ &= B_n\cos\theta_n - A_n\sin\theta_n + \mathrm{i}(B_n\sin\theta_n + A_n\cos\theta_n), \\ \Im\left(z\mathrm{e}^{\mathrm{i}\theta_n}\right) &= A_n\cos\theta_n + B_n\sin\theta_n. \end{aligned}$$

The complex number z can be re-written with its Cartesian coordinates as $(B_n, A_n) = (|z|\cos\gamma_n, |z|\sin\gamma_n)$, hence $\tan\gamma_n = A_n/B_n$, and $z = |z|\mathrm{e}^{\mathrm{i}\gamma_n}$, with $|z| = \sqrt{A_n^2 + B_n^2}$. Thus,

$$z\mathrm{e}^{\mathrm{i}\theta_n} = |z|\mathrm{e}^{\mathrm{i}(\theta_n+\gamma_n)},$$

and

$$\Im\left(z\mathrm{e}^{\mathrm{i}\theta_n}\right) = \sqrt{A_n^2 + B_n^2}\sin(\theta_n + \gamma_n) = A_n\cos\theta_n + B_n\sin\theta_n.$$

11.5 Energy

For the diffusion equation $u_t = Du_{xx}$, either

$$E_1 = \int_\Omega u(x,t)\mathrm{d}x,$$

or

$$E_2 = \tfrac{1}{2}\int_\Omega u^2(x,t)\mathrm{d}x,$$

has the interpretation of energy, depending on the physical context. Both of these are decreasing functions of time, since the general solution is

$$|u(x,t)| = \left|\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2 Dt/L^2}\right| \le |u(x,0)|.$$

In this section we formulate an energy for the wave equation and show that it is conserved.

To do this, we recall the discrete starting point for the wave equation. We took N identical particles arrayed on a line, connected together by identical springs. The equilibrium position of the i^{th} particle is $x_i = i\Delta x$, and the departure from equilibrium is small and equal to y_i. The potential energy of such a system is

$$\mathcal{U}(y_1, \cdots, y_N) = \tfrac{1}{2}k\sum_{i=0}^{N}\left(y_{i+1} - y_i\right)^2, \qquad y_0 = y_{N+1} = 0.$$

At interior points ($i = 1, 2, \cdots, N$), Newton's law gives

$$m\frac{d^2 y_i}{dt^2} = -\frac{\partial \mathcal{U}}{\partial y_i} = k\left[(y_{i+1} - y_i) - (y_i - y_{i-1})\right].$$

This is an equation of the type

$$m\frac{d^2 \boldsymbol{y}}{dt^2} = -\nabla_y \mathcal{U}(\boldsymbol{y}), \qquad \boldsymbol{y} = (0, y_1, \cdots, y_N, 0)^T.$$

If we take the dot product of this equation with $d\boldsymbol{y}/dt$ we obtain

$$\begin{aligned} m\frac{d\boldsymbol{y}}{dt}\cdot\frac{d^2\boldsymbol{y}}{dt^2} &= -\frac{d\boldsymbol{y}}{dt}\cdot\nabla_y\mathcal{U}(\boldsymbol{y}), \\ m\frac{d}{dt}\left(\frac{d\boldsymbol{y}}{dt}\right)^2 &= -\frac{d}{dt}\mathcal{U}(\boldsymbol{y}), \end{aligned}$$

or

$$\tfrac{1}{2}m\left(\frac{d\boldsymbol{y}}{dt}\right)^2 + \mathcal{U}(\boldsymbol{y}) = E = \text{Const.}$$

In other words,

$$\tfrac{1}{2}m\sum_{i=1}^{N}\left(\frac{dy_i}{dt}\right)^2 + \tfrac{1}{2}k\sum_{i=0}^{N}(y_{i+1} - y_i)^2 = E = \text{Const.}, \qquad y_0 = y_{N+1} = 0.$$

As before, let $m = \rho\Delta x$ and let $k\Delta x = T = \text{Const.}$. Hence, $k = T/\Delta x$ and

$$\tfrac{1}{2}\rho\sum_{i=1}^{N}\Delta x\left(\frac{dy_i}{dt}\right)^2 + \tfrac{1}{2}T\sum_{i=0}^{N}\Delta x\frac{(y_{i+1} - y_i)^2}{(\Delta x)^2} = E.$$

Now take $\Delta x \to 0$. The sums become Riemann integrals and the finite differences become derivatives.

$$\tfrac{1}{2}\int_\Omega \mathrm{d}x\,\rho\left(\frac{\partial u}{\partial t}\right)^2 + \tfrac{1}{2}T\int_\Omega \mathrm{d}x\,\left(\frac{\partial u}{\partial x}\right)^2 = E.$$

Thus, our candidate for conserved pseudo-energy is

$$\mathcal{E} := \int_\Omega \left[\frac{1}{c^2}\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right]\mathrm{d}x.$$

(I call it a pseudo-energy because strictly speaking, it does not have dimensions of energy.) Now finally, let's double check that the wave equation $c^{-2}\partial_{tt}u = \partial_{xx}$ with the zero BCs conserves the pseudo-energy:

$$
\begin{aligned}
\frac{d\mathcal{E}}{dt} &= \int_\Omega \left[\frac{1}{c^2}\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial t \partial x}\right], \\
&= \int_\Omega \left[\frac{1}{c^2}\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial x}\frac{\partial}{\partial x}\frac{\partial u}{\partial t}\right] \mathrm{d}x, \\
&= \int_\Omega \left[\frac{1}{c^2}\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right) - \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2}\right] \mathrm{d}x, \\
&= \int_\Omega \left[\frac{1}{c^2}\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} - \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2}\right] \mathrm{d}x + \left(\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right)_0^L, \\
&= \int_\Omega \left[\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2}\right]\frac{\partial u}{\partial t}\mathrm{d}x + \underbrace{\frac{\partial u(L,t)}{\partial t}}_{=0}\left(\frac{\partial u}{\partial x}\right)_{x=L} - \underbrace{\frac{\partial u(0,t)}{\partial t}}_{=0}\left(\frac{\partial u}{\partial x}\right)_{x=0}, \\
&= 0 - 0.
\end{aligned}
$$