ACM 40990 / ACM41030
A note on Theorem 2.10

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In class on 01/02/2024 we looked at Theorem 2.10. In particular:

__Theorem__ : f has a unique minimizer if and only if B is strictly P.D.

__Proof__ : Assume that B is P.D. Then, B is invertible, so $\underline{a}$ is in the range of B, so let $\underline{x}$ solve $B\underline{x} = -\underline{a}$.

Consider :

$$f(\underline{x}+\underline{w}) = f(\underline{x}) + \tfrac{1}{2}\langle \underline{w}, B\underline{w}\rangle$$

But B is P.D. so $\langle \underline{w}, B\underline{w}\rangle > 0 \ \forall \underline{w} \neq 0$.

Hence $f(\underline{x}+\underline{w}) > f(\underline{x}) \ \forall \ \underline{w} \ (\neq 0) \in \mathbb{R}^n$.

So $\underline{x}$ is the unique global minimizer.

This was all straightforward. We also looked at the statement the other way around:

For the other way around, suppose f has a unique global minimizer (call it $\underline{x}$). We use a proof by contradiction: assume that B is not positive-definite.

The strategy here was to do a proof by contradiction. But there was a small gap in the proof. Here, I fill in the gap. Hence, suppose as stated, and for contradiction, that B is not positive-defintie. Then, there is some non-zero direction $\boldsymbol{w}$ such that:

$$\langle \boldsymbol{w}, B\boldsymbol{w} \rangle \leq 0.$$

We look at two cases:

- Case 1: If $\langle \boldsymbol{w}, B\boldsymbol{w} \rangle < 0$. In this case, a contradiction ensues, since we would have $f(\boldsymbol{x}+\boldsymbol{w}) < f(\boldsymbol{x})$, for $\boldsymbol{x} = -B^{-1}\boldsymbol{a}$.
- Case 2: If $\langle \boldsymbol{w}, B\boldsymbol{w} \rangle = 0$. This is the case that is looked at in the lecture notes, so we can jump right back into the lecture notes at this point and finish the proof.

Then, we can find a non-zero vector $\underline{w}$ such that $B\underline{w} = 0$. Then,

$$f(\underline{x}+\underline{w}) = f(\underline{x}) + \cancel{\tfrac{1}{2}\langle \underline{w}, B\underline{w} \rangle}$$
$$= f(\underline{x}).$$

Hence, $\underline{x}+\underline{w}$ is also a minimizer. This contradicts uniqueness. Hence, the only way to have a unique global minimizer is if B is strictly P.D.