Mechanics and Special Relativity (ACM10030)
Assignment 4

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In these questions, you may use the following conversion factor relating the electron-volt to Joules: $1\,\mathrm{eV} = 1.60217646 \times 10^{-19}\,\mathrm{Joules}$, where the Joule is the SI unit of energy, $\mathrm{J} = \mathrm{kg}\,\mathrm{m}^2/\mathrm{s}^2$. Furthermore, you may use the fact that the rest energy of an electron is $m_\mathrm{e}c^2 = 0.511\,\mathrm{MeV}$.

1. **Satellite motion, Galilean physics** A satellite of mass m is in circular orbit about the earth. The radius of the orbit is r_0 and the mass of the earth is M_e.
   (a) Find the total mechanical energy of the satellite.
   (b) Now suppose that the satellite moves in the extreme upper atmosphere of the earth where it is retarded by a constant but small friction force f. The satellite will slowly spiral towards the earth. Since the friction force is weak, the change in radius will be very slow. Therefore, we assume that at any instant the satellite is effectively in circular orbit of average radius r. Find the approximate change in radius per revolution of the satellite, Δr.
   (c) Find the approximate change in the kinetic energy of the satellite per revolution, ΔK.

(a) We use the ellipse formula:

$$E = -\frac{GM_\mathrm{e}m}{2a},$$

where a is the semimajor axis. Now a circle is a degerate ellipse, whose semimajor and semiminor axes are equal. Thus, the energy of the orbit is

$$E = -\frac{GM_\mathrm{e}m}{2r_0}.$$

(b) We use the power equation:

$$\frac{dE}{dt} = \boldsymbol{f} \cdot \boldsymbol{v},$$

where $\boldsymbol{f}$ is the frictional force and $\boldsymbol{v}$ is the velocity. Now friction opposes motion, so $\boldsymbol{f} = -f\,(\boldsymbol{v}/|\boldsymbol{v}|)$, and $\boldsymbol{f}\cdot\boldsymbol{v} = -fv$. Find dE/dt:

$$\frac{dE}{dt} = -\frac{d}{dt}\left(GM_\mathrm{e}m/2r\right) = \frac{GM_\mathrm{e}m}{2r^2}\frac{dr}{dt} = -fv.$$

Hence,

$$\frac{GM_\mathrm{e}m}{2r^2}\frac{dr}{dt} = -fv.$$

But $dr/dt = (dr/d\theta)\,\dot{\theta}$, and

$$\frac{GM_\mathrm{e}m}{2r^2}\frac{dr}{d\theta} = -f\left(v/\dot{\theta}\right).$$

But the motion such that it is always circular, and $v = v_\theta = r\dot{\theta}$. Hence,

$$\frac{GM_\mathrm{e}m}{2r^2}\frac{dr}{d\theta} = -fr.$$

Thus,

$$\frac{dr}{d\theta} = -fr\left(\frac{2r^2}{GM_\mathrm{e}m}\right) = -\frac{2fr^3}{GM_\mathrm{e}m}.$$

Integrating over a revolution gives Δr:

$$\Delta r = -\frac{2f}{GM_\mathrm{e}m}\int_0^{2\pi} r\,(\theta)^3\,\mathrm{d}\theta.$$

But the radius varies very slowly with θ, because $dr/\theta \propto f$. Therefore, we take the integrand out from under the integral sign, and

$$\Delta r = -\frac{2r^3 f}{GM_\mathrm{e}m}\times 2\pi = -\frac{4\pi r^3 f}{GM_\mathrm{e}m}.$$

Not surprisingly, the change in radius is negative, and the satellite spirals towards the earth.

(c) Start with the energy partition

$$-\frac{GM_\mathrm{e}m}{2r} = K - \frac{GM_\mathrm{e}m}{r},$$

hence

$$K = \frac{GM_\mathrm{e}m}{r} > 0.$$

Thus,

$$\frac{dK}{d\theta} = -\frac{GM_\mathrm{e}m}{2r^2}\frac{dr}{d\theta}.$$

But $dr/d\theta = -2fr^3/\left(GM_\mathrm{e}m\right)$, and

$$\frac{dK}{d\theta} = \frac{GM_\mathrm{e}m}{2r^2}\frac{2fr^3}{(GM_\mathrm{e}m)}.$$

Effecting the cancellations, this is

$$\frac{dK}{d\theta} = fr,$$

and the kinetic energy INCREASES, albeit slowly (because $dK/dr \propto f$). Integrating over a revolution and taking $r(\theta)$ outside the integrand gives

$$\Delta K = 2\pi fr.$$

2. **The Lorentz transformations**

(a) An observer in frame S' is moving to the right at speed $V = 0.600c$ away from a stationary observer in frame S. The observer in S' measures the speed v' of a particle moving to the right away from her. What speed v does the observer in S measure the particle if $v' = 0.900c$?

(b) A pursuit spacecraft from the planet Tatooine is attempting to catch up with a Trade Federation cruiser. As measured by an observer on Tatooine, the cruiser is travelling away from the planet with a speed $0.600c$. The pursuit ship is travelling at a speed $0.800c$ relative to Tatooine, in the same direction as the cruiser. What is the speed of the cruiser relative to the pursuit ship?

(c) Two particles are created in a high-energy accelerator and move off in opposite directions: one to the left, and one to the right. The speed of one particle as measured in the lab is $0.650c$ and the speed of each particle relative to the other is $0.950c$. What is the speed of the second particle, as measured in the lab?

(a) The LT formula is

$$v' = \frac{v - V}{1 - \frac{vV}{c^2}},$$

where v is the velocity of the object in frame S, v' is the velocity of the object in frame S', and V is the velocity of frame S'. w.r.t. frame S. Inverting this formula gives

$$v = \frac{v' + V}{1 + \frac{v'V}{c^2}}.$$

For us, $v' = 0.900c$ and $V = 0.600c$, the speed of the frame. Hence,

$$v = \frac{0.6 + 0.9}{1 + 0.6 \times 0.9} = 0.974c.$$

(b) In this problem, let the planet corresponds to frame S, and let the pursuit ship correspond to frame S'. Frame S' has a velocity $V = 0.800c$, and finally, the cruiser corresponds to the third object, with velocity $v = 0.600c$. We must compute v', the velocity of the cruiser in the frame of the pursuit ship. Using

$$v' = \frac{v - V}{1 - \frac{vV}{c^2}},$$

this is

$$v' = \frac{0.6 - 0.8}{1 - 0.6 \times 0.8} = -0.385c,$$

and this is negative, meaning the pursuit ship will catch the cruiser.

(c) In this problem, let the speed of the particle B w.r.t. the lab be $v = 0.650c$ towards the right. Let the speed of particle A be V (with an implied sign). This is also the velocity of frame S' w.r.t. frame S, the lab frame. We are given $v' = 0.960c$ and we must find V. We take the standard LT formula for v' and invert for V:

$$V = \frac{v - v'}{1 - \frac{vv'}{c^2}}.$$

Hence,

$$V = \frac{0.65 - 0.95}{1 - 0.65 \times 0.95} = -0.784c$$

and particle A travels to the left in the lab, as expected.

3. **Energy**

(a) What is the speed of a particle if its kinetic energy is 1.0% larger than $mv^2/2$? Hint: Use the Binomial Theorem.

(b) The kinetic energy of a certain electron is 0.520-MeV. To create x-rays (high-energy photons), the electron travels down a tube and hits a target. When it arrives at the target, what is its kinetic energy in eV? What is its total energy? What is its speed? What is the speed of the electron, computed (incorrectly) from Newtonian mechanics?

(a) In this problem, the particle is just barely relativistic, and this suggests using the binomial expansion. We introduce the parameter $\beta := v/c$. The KE is

$$\left[\left(1 - \beta^2\right)^{-1/2} - 1\right] mc^2 = \tfrac{1}{2} m \left(1 + \delta\right) v^2,$$

The binomial theorem with $x = -\beta^2$ and $n = -1/2$ gives

$$\left(1 - \beta^2\right)^{-1/2} = 1 + \tfrac{1}{2}\beta^2 + \tfrac{3}{8}\beta^4 + \tfrac{5}{16}\beta^6 + \cdots .$$

where $\delta = 0.03$. Taking only up to the fourth power gives

$$\left(1 + \tfrac{1}{2}\beta^2 + \tfrac{3}{8}\beta^4\right) - 1 = \tfrac{1}{2}\beta^2 + \tfrac{1}{2}\delta\beta^2.$$

Effecting the cancellations gives

$$\beta = \sqrt{4\delta/3}.$$

Plugging in $\delta = 0.01$ gives $\beta = 0.1155$, or $\beta = 0.1$ correct to one SF.
We must check that we have not lost any precision in truncating the binomial expansion. So, let's go to the next order in the expansion:

$$\left(1 + \tfrac{1}{2}\beta^2 + \tfrac{3}{8}\beta^4 + \tfrac{5}{16}\beta^6\right) - 1 = \tfrac{1}{2}\beta^2 + \tfrac{1}{2}\delta\beta^2.$$

Effecting cancellations gives

$$\tfrac{5}{16}\beta^4 + \tfrac{3}{8}\beta^2 - \tfrac{1}{2}\delta = 0.$$

Hence,

$$\beta^2 = \frac{-\frac{3}{8} \pm \sqrt{\frac{3^2}{8^2} + 4\delta\frac{5}{16}\frac{1}{2}}}{2\frac{5}{16}} = 0.0132$$

or $\beta = 0.1148$, and $\beta = 0.1$, correct to one SF. Hence, for accuracy to only one SF, we are correct to truncate after the β^4 term in the binomial expansion.

(b) Re-scaling the energy gives $K = 5.20 \times 10^5\,\mathrm{eV}$.
The rest energy is $mc^2 = 5.11 \times 10^5\,\mathrm{eV}$, so the total energy is

$$E = (5.20 + 5.11) \times 10^5\,\mathrm{eV} = 10.31 \times 10^5\,\mathrm{eV}.$$

Now $E = \gamma mc^2$, so

$$\gamma = \frac{E}{mc^2} = \frac{10.31}{5.11} := \rho.$$

Inverting gives

$$v = c\sqrt{1 - \frac{1}{\rho^2}} = c\sqrt{1 - \left(\frac{5.11}{10.31}\right)^2} = 0.8685c = 2.61 \times 10^8\,\mathrm{m/s}.$$

Computing INCORRECTLY with the NR formula gives

$$\tfrac{1}{2}mv^2 = 0.520\,\mathrm{MeV},$$

or

$$\tfrac{1}{2}\frac{mv^2}{mc^2} = \tfrac{1}{2}\frac{v^2}{c^2} = \frac{0.520}{0.511},$$

hence

$$v = c\sqrt{2 \times 5.20/5.11} = 1.43c.$$

4. **Scattering experiments**

(a) A photon with energy E is emitted by an atom with mass m, which recoils in the opposite direction. Assuming that the motion of the atom can be treated nonrelativistically, compute the recoil speed of the atom. From this result, show that the recoil speed is much smaller than c whenever E is much smaller than the rest energy mc^2 of the atom.

(b) Two pions π^+ and π^- collide and produce a neutral kaon. If the event is a head-on collision in which the pions have velocities v_0 and $-v_0/2$ in the laboratory frame, what is the mass of the kaon in terms of the velocity v_0 and the pion mass m_π? Find a numerical result (with m_π still undetermined) if $v_0 = 0.95c$.

(a) Using NR conservation of momentum,

$$mv = p_{\text{photon}} = \frac{E}{c}.$$

Hence,

$$v = \frac{E}{mc}.$$

This is the final answer. Re-arranging the balance law gives

$$\frac{v}{c} = \frac{E}{mc^2}.$$

Hence, the NR assumption for the recoil is correct when $E \ll mc^2$.

(b) The energy balance gives

$$m_\pi c^2 \left[\gamma\left(v_0\right) + \gamma\left(\tfrac{1}{2}v_0\right)\right] = \gamma\left(v\right) m_K c^2,$$

where v is the Kaon velocity. Cancellation yields

$$m_\pi \left[\gamma\left(v_0\right) + \gamma\left(\tfrac{1}{2}v_0\right)\right] = m_K \gamma\left(v\right). \tag{1}$$

It remains to compute $\gamma\left(v\right)$. Conservation of momentum yields

$$m_\pi \left[\gamma\left(v_0\right) - \tfrac{1}{2}\gamma\left(\tfrac{1}{2}v_0\right)\right] v_0 = m_K \gamma\left(v\right) v.$$

Divide out by v:

$$\frac{m_\pi \left[\gamma\left(v_0\right) - \tfrac{1}{2}\gamma\left(\tfrac{1}{2}v_0\right)\right] v_0}{v} = m_K \gamma\left(v\right). \tag{2}$$

Equate Eqs. (1)–(2):

$$\frac{m_\pi \left[\gamma\left(v_0\right) - \tfrac{1}{2}\gamma\left(\tfrac{1}{2}v_0\right)\right] v_0}{v} = m_\pi \left[\gamma\left(v_0\right) + \gamma\left(\tfrac{1}{2}v_0\right)\right].$$

Hence,

$$v = v_0 \frac{\gamma\left(v_0\right) - \tfrac{1}{2}\gamma\left(\tfrac{1}{2}v_0\right)}{\gamma\left(v_0\right) + \gamma\left(\tfrac{1}{2}v_0\right)}$$

The final answer requires plugging in this form of v into the equation

$$m_K = \frac{m_\pi \left[\gamma\left(v_0\right) + \gamma\left(\tfrac{1}{2}v_0\right)\right]}{\gamma\left(v\right)}.$$

Numerical values: Let $v_0 = 0.95c$ to obtain

$$\frac{\gamma\left(v_0\right) - \frac{1}{2}\gamma\left(\frac{1}{2}v_0\right)}{\gamma\left(v_0\right) + \gamma\left(\frac{1}{2}v_0\right)} = 0.6071,$$

and

$$\gamma\left(v_0\right) + \gamma\left(\tfrac{1}{2}v_0\right) = 4.3389.$$

Hence,

$$v = 0.95 \times 0.6071c = 0.5768c,$$

$$\gamma\left(v\right) = 1.2241$$

and finally,

$$m_K = \frac{m_\pi\left[\gamma\left(v_0\right) + \gamma\left(v_0/2\right)\right]}{\gamma\left(v\right)} = m_\pi\left(\frac{4.3389}{1.2241}\right) \approx 3.54m_\pi.$$