Mechanics and Special Relativity (MAPH10030)
Assignment 4

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In these questions, you may use the following conversion factor relating the electron-volt to Joules: $1\,\mathrm{eV} = 1.60217646 \times 10^{-19}\,\mathrm{Joules}$, where the Joule is the SI unit of energy, $\mathrm{J} = \mathrm{kg}\,\mathrm{m}^2/\mathrm{s}^2$. Furthermore, you may use the fact that the rest energy of an electron is $m_{\mathrm{e}}c^2 = 0.511\,\mathrm{MeV}$.

1. **The Lorentz transformations**

   (a) An observer in frame S' is moving to the right at speed $V = 0.600c$ away from a stationary observer in frame S. The observer in S' measures the speed v' of a particle moving to the right away from her. What speed v does the observer in S measure the particle if $v' = 0.900c$? [2 points]

   The LT formula is
   $$v' = \frac{v - V}{1 - \frac{vV}{c^2}},$$
   where v is the velocity of the object in frame S, v' is the velocity of the object in frame S', and V is the velocity of frame S'. w.r.t. frame S. Inverting this formula gives
   $$v = \frac{v' + V}{1 + \frac{v'V}{c^2}}.$$
   For us, $v' = 0.900c$ and $V = 0.600c$, the speed of the frame. Hence,
   $$v = \frac{0.6 + 0.9}{1 + 0.6 \times 0.9} = 0.974c.$$

   (b) A pursuit spacecraft from the planet Tatooine is attempting to catch up with a Trade Federation cruiser. As measured by an observer on Tatooine, the cruiser is travelling away from the planet with a speed $0.600c$. The pursuit ship is travelling at a speed $0.800c$ relative to Tatooine, in the same direction as the cruiser. What is the speed of the cruiser relative to the pursuit ship? [2 points]

   In this problem, the planet corresponds to frame S, and the pursuit ship corresponds to frame S'. Frame S' has a velocity $V = 0.800c$, while finally, the crusier corresponds to the third object, with velocity $v = 0.600c$. We must compute v', the velocity of the cruiser in the frame of the pursuit ship. Using

$$v' = \frac{v - V}{1 - \frac{vV}{c^2}},$$

this is

$$v' = \frac{0.6 - 0.8}{1 - 0.6 \times 0.8} = -0.385c,$$

and this is negative, meaning the pursuit ship will catch the cruiser.

(c) Two particles are created in a high-energy acclelerator and move off in opposite directions. The speed of one particle as measured in the lab is $0.650c$ and the speed of each particle realtive to the other is $0.950c$. What is the speed of the second particle, as measured in the lab? [2 points]

In this problem, let the speed of the particle B w.r.t. the lab be $v = 0.650c$ towards the right. Let the speed of particle A be V (with an implied sign). This is also the velocity of frame S' w.r.t. frame S, the lab frame. We are given $v' = 0.960c$ and we must find V. We take the standard LT formula for v' and invert for V:

$$V = \frac{v - v'}{1 - \frac{vv'}{c^2}}.$$

Hence,

$$V = \frac{0.65 - 0.95}{1 - 0.65 \times 0.95} - 0.784c$$

and particle A travels to the left in the lab, as expected.

2. **Energy**

(a) What is the speed of a particle if its kinetic energy is 2.0% larger than $mv^2/2$? [2 points]

In this problem, the particle is just barely relativistic, and this suggests using the binomial expansion. We introduce the parameter $\beta := v/c$. The KE is

$$\left[\left(1 - \beta^2\right)^{-1/2} - 1\right] mc^2 = \tfrac{1}{2}m\left(1 + \delta\right)v^2,$$

The binomial theorme with $x = -\beta^2$ and $n = -1/2$ gives

$$\left(1 - \beta^2\right)^{-1/2} = 1 + \tfrac{1}{2}\beta^2 + \tfrac{3}{8}\beta^4 + \tfrac{5}{16}\beta^6 + \cdots.$$

where $\delta = 0.02$. Taking only up to the fourth power gives

$$1 + \tfrac{1}{2}\beta^2 + \tfrac{3}{8}\beta^4 = \tfrac{1}{2}\beta^2 + \tfrac{1}{2}\delta\beta^2.$$

Effecting the cancellations gives

$$\beta = \sqrt{4\delta/3}.$$

Plugging in $\delta = 0.02$ gives $\beta = 0.1633$, which to two SFs is $\beta = 0.16$.

We must check that we have not lost any precision in truncating the binomial expansion. So, let's go to the next order in the expansion:

$$1 + \tfrac{1}{2}\beta^2 + \tfrac{3}{8}\beta^4 + \tfrac{5}{16}\beta^6 = \tfrac{1}{2}\beta^2 + \tfrac{1}{2}\delta\beta^2.$$

Effecting cancellations gives

$$\tfrac{5}{16}\beta^4 + \tfrac{3}{8}\beta^2 - \tfrac{1}{2}\delta = 0.$$

Hence,

$$\beta^2 = \frac{-\frac{3}{8} \pm \sqrt{\frac{3^2}{8^2} + 4\delta\frac{5}{16}\frac{1}{2}}}{2\frac{5}{16}} = 0.0261$$

or $\beta = 0.16$, to two SFs. Hence, we are correct to truncate after the β^4 term in the binomial expansion.

(b) The kinetic energy of a certain electron is 0.420-MeV. To create x-rays (high-energy photons), the electron travels down a tube and hits a target. When it arrives at the target, what is its kinetic energy in eV? What is its total energy? What is its speed? What is the speed of the electron, computed (incorrectly) from Newtonian mechanics? [2 points]

Re-scaling the energy gives $K = 4.20 \times 10^5\,\mathrm{eV}$.

The rest energy is $mc^2 = 5.11 \times 10^5\,\mathrm{eV}$, so the total energy is $E = 0.931\,\mathrm{MeV} = 9.31 \times 10^5\,\mathrm{eV}$.

Now $E = \gamma mc^2$, so

$$\gamma = \frac{E}{mc^2} = \frac{0.931}{0.511} := \rho.$$

Inverting gives

$$v = c\sqrt{1 - \frac{1}{\rho^2}} = c\sqrt{1 - \left(\frac{0.511}{0.931}\right)^2} = 0.836c = 2.51 \times 10^8\,\mathrm{m/s}.$$

Computing INCORRECTLY with the NR formula gives

$$\tfrac{1}{2}mv^2 = 0.420\,\mathrm{MeV},$$

or

$$\tfrac{1}{2}\frac{mv^2}{mc^2} = \tfrac{1}{2}\frac{v^2}{c^2} = \frac{0.420}{0.511},$$

hence

$$v = c\sqrt{2 \times 0.420/0.511} = 1.282c \approx 3.85 \times 10^8\,\mathrm{m/s}.$$

3. **Scattering experiments**

(a) A photon with energy E is emitted by an atom with mass m, which recoils in the opposite direction. Assuming that the motion of the atom can be treated nonrelativistically, compute the recoil speed of the atom. From this result, show that the recoil speed is much smaller than c whenever E is much smaller than the rest energy mc^2 of the atom [2 points].

Using NR conservation of MOMENTUM,

$$mv = p_{\text{photon}} = \frac{E}{c}.$$

Hence,

$$v = \frac{E}{mc}.$$

This is the final answer. Re-arranging the balance law gives

$$\frac{v}{c} = \frac{E}{mc^2}.$$

Hence, the NR assumption for the recoil is correct when $E \ll mc^2$.

Note that conservation of ENERGY tells us nothing about the recoil speed. It can, however, tell us something about the excitation energy Q of the atom:

$$Q = \tfrac{1}{2}mv^2 + E,$$

which is the statement that the atom de-excites by the emission of a photon. Using Eq. (**??**), the excitation energy is

$$Q = E\left(1 + \tfrac{1}{2}\frac{E}{mc^2}\right).$$

If $E \ll mc^2$, then $Q \approx E$, and the excitation energy Q gives rise to a photon with frequency $h\nu \approx Q$. This approximate relation is the one given in the standard description of quantum mechanics (e.g. University Physics, Ch. 40), but is clearly only valid in the NR limit $E \ll mc^2$.

(b) In Compton scattering, what is the maximum possible wavelength shift? [2 points]

The formula for Compton scattering is

$$\Delta\lambda := \lambda' - \lambda = \frac{h}{mc}\left(1 - \cos\phi\right).$$

If we regard $\Delta\lambda$ as a function of the scattering angle ϕ, then $\Delta\lambda$ is maximal when $\cos\phi = -1$, that is, when $\phi = \pi$. Then,

$$\Delta\lambda_{\max} = \frac{2h}{mc}.$$

(c) Two pions π^+ and π^- collide and produce a neutral kaon. If the event is a head-on collision in which the pions have velocities v_0 and $-v_0/2$ in the laboratory frame what is the mass of the kaon in terms of the velocity v_0 and the pion mass m_π? Find a numerical result (with m_π still undetermined) if $v_0 = 0.99c$. [6 points]

The energy balance gives

$$m_\pi c^2\left[\gamma\left(v_0\right) + \gamma\left(\tfrac{1}{2}v_0\right)\right] = \gamma\left(v\right) m_K c^2,$$

where v is the Kaon velocity. Cancellation yields

$$m_\pi \left[\gamma\left(v_0\right) + \gamma\left(\tfrac{1}{2}v_0\right)\right] = m_K \gamma\left(v\right). \tag{1}$$

It remains to compute $\gamma\left(v\right)$. Conservation of momentum yields

$$m_\pi \left[\gamma\left(v_0\right) - \tfrac{1}{2}\gamma\left(\tfrac{1}{2}v_0\right)\right] v_0 = m_K \gamma\left(v\right) v.$$

Divide out by v:

$$\frac{m_\pi \left[\gamma\left(v_0\right) - \tfrac{1}{2}\gamma\left(\tfrac{1}{2}v_0\right)\right] v_0}{v} = m_K \gamma\left(v\right). \tag{2}$$

Equate Eqs. (1)–(2):

$$\frac{m_\pi \left[\gamma\left(v_0\right) - \tfrac{1}{2}\gamma\left(\tfrac{1}{2}v_0\right)\right] v_0}{v} = m_\pi \left[\gamma\left(v_0\right) + \gamma\left(\tfrac{1}{2}v_0\right)\right].$$

Hence,

$$v = v_0 \frac{\gamma\left(v_0\right) - \tfrac{1}{2}\gamma\left(\tfrac{1}{2}v_0\right)}{\gamma\left(v_0\right) + \gamma\left(\tfrac{1}{2}v_0\right)}$$

The final answer requires plugging in this form of v into the equation

$$m_K = \frac{m_\pi \left[\gamma\left(v_0\right) + \gamma\left(\tfrac{1}{2}v_0\right)\right]}{\gamma\left(v\right)}.$$

Numerical values: Letting $v_0 = 0.95c$ gives

$$\frac{\gamma\left(v_0\right) - \tfrac{1}{2}\gamma\left(\tfrac{1}{2}v_0\right)}{\gamma\left(v_0\right) + \gamma\left(\tfrac{1}{2}v_0\right)} = 0.6071,$$

and

$$\gamma\left(v_0\right) + \gamma\left(\tfrac{1}{2}v_0\right) = 4.3389.$$

Hence,

$$v = 0.95 \times 0.6071c = 0.5768c,$$

$$\gamma\left(v\right) = 1.2241$$

and finally,

$$m_K = \frac{m_\pi \left[\gamma\left(v_0\right) + \gamma\left(v_0/2\right)\right]}{\gamma\left(v\right)} = m_\pi \left(\frac{4.3389}{1.2241}\right) \approx 3.54 m_\pi.$$

4. **Bonus question** [Top up for a maximum of 5 points] Two events observed in a frame of reference S have positions and times given by (x_1, t_1) and (x_2, t_2), respectively.

(a) Frame S' moves along the x-axis just fast enough that the two events occur at the same position in S'. Show that in S', the time interval $\Delta t'$ between the two events is given by

$$\Delta t' = \sqrt{(\Delta t)^2 - (\Delta x/c)^2}, \qquad \Delta x = x_2 - x_1, \Delta t = t_2 - t_1.$$

We have the LTs

$$\begin{aligned} x' &= \gamma\left(x - Vt\right), \\ t' &= \gamma\left(t - \frac{Vx}{c^2}\right). \end{aligned}$$

Because they are linear, we can take increments in a linear fashion:

$$\begin{aligned} \Delta x' &= \gamma\left(\Delta x - V\Delta t\right), \\ \Delta t' &= \gamma\left(\Delta t - \frac{V\Delta x}{c^2}\right). \end{aligned}$$

By assumption, in frame S', $\Delta x' = 0$, hence $\Delta x = V\Delta t$. Thus,

$$V = \frac{\Delta x}{\Delta t} \implies$$

$$\gamma = \frac{1}{\sqrt{1-(V^2/c^2)}} = \frac{c}{\sqrt{c^2 - \Delta x^2/\Delta t^2}} = \frac{c\Delta t}{\sqrt{c^2\Delta t^2 - \Delta x^2}}.$$

Plugging this into

$$\Delta t' = \gamma\left(\Delta t - \frac{V\Delta x}{c^2}\right)$$

gives

$$\Delta t' = \frac{c\Delta t}{\sqrt{c^2\Delta t^2 - \Delta x^2}}\left(\Delta t - \frac{\Delta x^2}{c^2\Delta t}\right).$$

Tidying up gives

$$\Delta t' = \sqrt{\Delta t^2 - (\Delta x/c)^2},$$

as required.

Hence, show that if $\Delta x > c\Delta t$, there is NO frame S' in which the two events occur at the same point [2 points].

Clearly, if $\Delta x > c\Delta t$, then $\Delta t'$ is imaginary. This cannot be the case, since $\Delta t'$ is a time interval. Thus, our assumption that $\Delta x' = 0$ breaks down, and the frame S', wherein both events take place at the same space point, cannot exist.

(b) Show that if $\Delta x > c\Delta t$, there is a different frame of reference S' in which the two events occur SIMULTANEOUSLY. Find the distance between the two events in S'; express your answer in terms of Δx, Δt, and c. [3 points]

We do the same thing: We compute increments of space and time in the new frame S' and impose $\Delta t' = 0$. Thus, from the LTs, $\Delta t = V\Delta x/c^2$, and

$$V = c^2\frac{\Delta t}{\Delta x}.$$

The γ-factor is

$$\gamma = \frac{1}{\sqrt{1 - c^4 \Delta t^2 / \left(c \Delta x\right)^2}} = \frac{\Delta x}{\sqrt{\Delta x^2 - c^2 \Delta t^2}}.$$

Plugging this into

$$\Delta x' = \gamma \left(\Delta x - V \Delta t\right)$$

gives

$$\Delta x' = \frac{\Delta x}{\sqrt{\Delta x^2 - c^2 \Delta t^2}} \left(\Delta x - c^2 \frac{\Delta t^2}{\Delta x}\right) = \sqrt{\Delta x^2 - c^2 \Delta t^2},$$

and this exists as an interval in space provided $\Delta x > c\Delta t$.