# Mechanics and Special Relativity (ACM10030)
# Assignment 3

Issue Date: xx March 2010
Due Date: xx March 2010

*Questions 1 and 2 carry five marks each; Question 3 carries ten marks. In question 2, a numerical answer is required. You may use* $M_{\mathrm{e}} = 5.97 \times 10^{24}\,\mathrm{kg}$. *In the other questions, a symbolic answer is fine.*

1. Refer to Fig. 1(a). A projectile of mass $m$ is fired from the surface of the earth at an angle $\alpha$ from the vertical. The initial speed $v_0$ is equal to $\sqrt{GM_{\mathrm{e}}/R_{\mathrm{e}}}$. How high does the projectile rise? Neglect air resistance and the earth's rotation.

   Hint: Do not try to solve for the orbit! Instead, use the conservation laws directly.

We use conservation of angular momentum. The initial angular momentum is

$$\boldsymbol{J}_{\mathrm{init}} = Rv_0 \sin\alpha \text{ into the page.}$$

The angular momentum at the maximum point is

$$\boldsymbol{J}_{\mathrm{top}} = r_{\mathrm{max}} v_1 \text{ into the page,}$$

and the velocity is purely tangent to the earth's surface at this point. Equating these quantities gives a formula for $v_1$ in terms of other things:

$$v_1 = Rv_0 \sin\alpha / r_{\mathrm{max}}.$$

Next, we use conservation of energy:

$$\tfrac{1}{2}v_0^2 - \frac{GM_{\mathrm{e}}}{R} = \tfrac{1}{2}v_1^2 - \frac{GM_{\mathrm{e}}}{r_{\mathrm{max}}}.$$

But $v_0^2 = GM_{\mathrm{e}}/R$. Hence,

$$-\tfrac{1}{2}\frac{GM_{\mathrm{e}}}{R} = \tfrac{1}{2}v_1^2 - \frac{GM_{\mathrm{e}}}{r_{\mathrm{max}}},$$

and

$$v_1^2 = \frac{R\sin^2\alpha GM_{\mathrm{e}}}{r_{\mathrm{max}}^2}.$$

So,

$$-\tfrac{1}{2}\frac{GM_{\rm e}}{R} = \tfrac{1}{2}R\sin^2\alpha GM_{\rm e}x^2 - GM_{\rm e}x, \qquad x = 1/r_{\rm max}.$$

Tidying up,

$$R\sin^2\alpha GM_{\rm e}x^2 - 2GM_{\rm e}x + (GM_{\rm e}/R) = 0,$$

with solution

$$x = \frac{GM_{\rm e} \pm \sqrt{G^2M_{\rm e}^2 - G^2M_{\rm e}^2\sin^2\alpha}}{R\sin^2\alpha GM_{\rm e}}.$$

This simplifies further:

$$x = \frac{1 \pm \sqrt{1-\sin^2\alpha}}{R\sin^2\alpha} = \frac{1+\cos\alpha}{R\sin^2\alpha}.$$

Finally,

$$r_{\rm max} = \frac{R\sin^2\alpha}{1\pm\cos\alpha}.$$

But which sign to choose? Note that if $\alpha = 0$, then the quadratic becomes degenerate and has solution $x = r_{\rm max}^{-1} = 1/(2R)$. We would like our formula to possess this behaviour: $r_{\rm max} \to 2R$ as $\alpha \to 0$. This suggests taking the MINUS sign. For, as $\alpha \to 0$, $\sin^2\alpha \sim \alpha^2$, and $1-\cos\alpha \sim \alpha^2/2$. Thus, the final answer is

$$r_{\rm max} = \frac{R\sin^2\alpha}{1-\cos\alpha}.$$

2. A space vehicle is in circular orbit around the earth. The mass of the vehicle is $3,000\,\mathrm{kg}$ and the radius of the orbit is $2R_{\rm e} = 12,800\,\mathrm{km}$. It is desired to transfer the vehicle to a circular orbit or radius $4R_{\rm e}$.

   (a) What is the minimum energy expenditure required for the transfer?

   (b) An efficient way to accomplish the transfer is to use a semi-elliptical orbit (known as a Hohmann transfer orbit), shown in the figure. What velocity changes are required at the points of intersection, points $A$ and $B$ in Fig. 1(b).

(a) Since a circular orbit minimises the effective potential, the minimum energy required for the transfer is associated with a transfer to a second circular orbit. Now

$$E_1 = -\frac{GM_{\rm e}m}{4R_{\rm e}},$$

and

$$E_2 = -\frac{GM_{\rm e}m}{8R_{\rm e}}$$

and the minimum energy is

$$\Delta E = E_2 - E_1 = -\frac{GM_{\rm e}m}{8R_{\rm e}} + \frac{GM_{\rm e}m}{4R_{\rm e}} = \frac{GM_{\rm e}m}{8R_{\rm e}}.$$

We also need a numerical answer to three significant figures. We use

$$GM_{\rm e}m/R_{\rm e} = 1.8666e+11,$$

and obtain $\Delta E = 2.33e+10\,\mathrm{J}$.

(b) *The first transfer point:* This involves a transition from an circular to an elliptic orbit, of semimajor axis $2a = 6R_{\rm e}$.

$$\begin{aligned} E_1 &= -\frac{GM_{\rm e}m}{4R_{\rm e}} = \tfrac{1}{2}mv_1^2 - \frac{GM_{\rm e}m}{2R_{\rm e}}, \\ E_{1t} &= -\frac{GM_{\rm e}m}{6R_{\rm e}} = \tfrac{1}{2}mv_{1t}^2 - \frac{GM_{\rm e}m}{2R_{\rm e}}, \end{aligned}$$

Solving for the velocities, obtain

$$v_1 = \sqrt{\frac{GM_{\rm e}}{2R_{\rm e}}}, \qquad v_{1t} = \sqrt{\frac{2GM_{\rm e}}{3R_{\rm e}}}.$$

Hence,

$$\Delta v_1 = v_{1t} - v_1 = \sqrt{GM_{\rm e}/R_{\rm e}}\left(\sqrt{2/3} - \sqrt{1/2}\right).$$

Now

$$\sqrt{GM_{\rm e}/R_{\rm e}} = 7.8879e+03,$$

hence

$$\Delta v_1 = 863\,\mathrm{m/s}.$$

*The second transfer point:* This involves a transition from elliptic orbit of semimajor axis $2a = 6R_{\rm e}$, to a circular orbit, of radius $4R\mathrm{e}$.

$$\begin{aligned} E_{2t} &= -\frac{GM_{\rm e}m}{6R_{\rm e}} = \tfrac{1}{2}mv_{2t}^2 - \frac{GM_{\rm e}m}{4R_{\rm e}}, \\ E_2 &= -\frac{GM_{\rm e}m}{8R_{\rm e}} = \tfrac{1}{2}mv_2^2 - \frac{GM_{\rm e}m}{4R_{\rm e}}, \end{aligned}$$

Solving for the velocities, obtain

$$v_{2t} = \sqrt{\frac{GM_{\rm e}}{6R_{\rm e}}}, \qquad v_2 = \sqrt{\frac{GM_{\rm e}}{4R_{\rm e}}}.$$

Hence,

$$\Delta v_2 = v_2 - v_{2t} = \sqrt{GM_{\rm e}/R_{\rm e}}\left(\sqrt{1/4} - \sqrt{1/6}\right).$$

Now

$$\sqrt{GM_{\rm e}/R_{\rm e}} = 7.8879e+03,$$

hence

$$\Delta v_2 = 724\,\mathrm{m/s}.$$

3. Consider a particle of mass $m$ in two dimensions, experiencing a central force $\boldsymbol{F} = -k\boldsymbol{r}$, where $\boldsymbol{r}$ is the radius vector of the particle relative to the force centre, and in an inertial frame. There are two ways of solving for the motion of such a system. The first way is as to write down the equations of motion in Cartesian form,

$$m\ddot{x} + kx = 0, \qquad m\ddot{y} + ky = 0,$$

and observe that the answer is two uncoupled SHM's, $x = A_x \cos(\omega t + \varphi_x)$, $y = A_y \cos(\omega t + \varphi_y)$, where $A_x$ and $A_y$ are constants. This solution pair satisfies the generic conic-section equation

$$A\,(x/A_x)^2 + B\,(x/A_x)\,(y/A_y) + C\,(y/A_y)^2 + D\,(x/A_x) + E\,(y/A_y) + F = 0,$$

where

$$A = C = 1, \qquad D = E = 0,$$
$$B = -2\cos\theta, \qquad F = -\sin^2\theta, \qquad \theta = \varphi_x - \varphi_y.$$

Hence, $B^2 - 4AC = 4(\cos^2\theta - 1) < 0$, and the motion is an ellipse. This is the quick and easy answer. However, the assignment requires that you follow the class notes, and find the answer as follows:

(a) What is the angular momentum $\boldsymbol{J}$ for the particle relative to the force centre? Show that this is conserved.

(b) Write down the equations of motion in two dimensions, in polar coordinates.
Hint: $F_r = -kr$, $F_\theta = 0$.

(c) Reduce the system to a one-equation problem and identify the effective potential. Sketch the result.
Answer: $m\ddot{r} = -\left(d\mathcal{U}_{\text{eff}}/dr\right)$, $\mathcal{U}_{\text{eff}} = [J^2/(2mr^2)] + (kr^2/2)$.

(d) Using the class notes as a hint, find the shape of the orbit of the particle. Are the orbits always closed?
Hint: Reduce the orbit problem to the integral

$$\theta - \theta_0 = \frac{J}{\sqrt{2m}} \int^r \frac{\mathrm{d}s}{s^2 \sqrt{E - \frac{1}{2}\frac{J^2}{ms^2} - \frac{1}{2}ks^2}},$$

and explain why $E$ is always positive. Then solve for the integral using

$$\mathcal{I} = \int \frac{\mathrm{d}s}{s\sqrt{-Cs^4 + Bs^2 - A}} = \underbrace{-\tfrac{1}{2}\int \frac{\mathrm{d}u}{\sqrt{-Au^2 + Bu - C}}}_{u=1/s^2} =$$
$$\tfrac{1}{2}\frac{1}{\sqrt{A}} \sin^{-1}\left[\frac{-2Au + B}{\sqrt{B^2 - 4AC}}\right],$$

where $A$, $B$, and $C$ are positive constants.

(e) Would a solar system governed by such a force law make sense? [0 points, but please think about this. The answer makes life on earth possible.]

(a) The angular momentum of the particle about the force centre is

$$\boldsymbol{J} = m\boldsymbol{r} \times \boldsymbol{v}, \qquad \boldsymbol{v} = \dot{\boldsymbol{r}}.$$

Newton's law for the particle is

$$m\ddot{\boldsymbol{r}} = -k\boldsymbol{r}.$$

The rate-of-change of angular momentum is

$$\frac{dJ}{dt} = \boldsymbol{r} \times \frac{d}{dt}(m\boldsymbol{v}) = \boldsymbol{r} \times (-k\boldsymbol{r}) = 0,$$

which we knew already because $\boldsymbol{F}$ is central.

(b) Because the angular momentum is conserved, the motion takes place in the plane defined by $\boldsymbol{r}(0)$ and $\boldsymbol{v}(0)$. Therefore we choose a coordinate system $(x, y)$ in this plane, whose origin is at the force centre. The potential for the central force $\boldsymbol{F} = -k\boldsymbol{r}$ is

$$\mathcal{U} = \tfrac{1}{2}k|\boldsymbol{r}|^2 = \tfrac{1}{2}kr^2,$$

hence $F_r = -\partial\mathcal{U}/\partial r = -kr$, and $F_\theta = -(1/r)(\partial\mathcal{U}/\partial\theta) = 0$. Newton's equations are thus

$$\begin{aligned} m\left(\ddot{r} - r\dot{\theta}^2\right) &= -\frac{\partial\mathcal{U}}{\partial r} = -kr, \\ m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) &= 0. \end{aligned}$$

(c) The EOM in the $\theta$ direction can be re-written as

$$\frac{m}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0,$$

hence $J = mr^2\dot{\theta}$ is conserved. This is the magnitude of the angular momentum, because $\boldsymbol{J} = mr\hat{\boldsymbol{r}} \times (\dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}) = mr^2\dot{\theta}\hat{\boldsymbol{z}}$, where $\hat{\boldsymbol{z}}$ is normal to the plane of motion. Therefore, $J/(mr^2) = \dot{\theta}$, and the centrifugal force in the radial equation becomes

$$mr\dot{\theta}^2 = mr\left(\frac{J}{mr^2}\right)^2 = \frac{J^2}{mr^3}.$$

The radial equation of motion is

$$\begin{aligned} m\ddot{r} &= \frac{J^2}{mr^3} - kr, \\ &= -\frac{\partial}{\partial r}\left(\tfrac{1}{2}\frac{J^2}{mr^2} + \tfrac{1}{2}kr^2\right), \\ &= -\frac{\partial\mathcal{U}_{\text{eff}}}{\partial r}, \qquad \mathcal{U}_{\text{eff}} = \tfrac{1}{2}\frac{J^2}{mr^2} + \tfrac{1}{2}kr^2. \end{aligned}$$

The effective potential is shown in Fig. 2.

(d) From the effective potential, we obtain the conserved energy

$$E = \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}\frac{J^2}{mr^2} + \tfrac{1}{2}kr^2 = \tfrac{1}{2}m\dot{r}^2 + \mathcal{U}_{\text{eff}}(r)\,.$$

The energy for this potential well is never negative, since $E \geq \mathcal{U}_{\text{eff}}(r) > 0$. The equation for $dr/dt$ is found from the conserved energy:

$$\frac{dr}{dt} = \sqrt{\frac{2}{m}}\sqrt{E - \mathcal{U}_{\text{eff}}(r)}.$$

We also know $d\theta/dt$ from angular-momentum conservation:

$$\frac{d\theta}{dt} = \frac{J}{mr^2}.$$

Dividing these equations one by the other and using the chain rule gives $dr/d\theta$:

$$\begin{aligned}
\frac{dr}{d\theta} &= \sqrt{\frac{2}{m}}\frac{mr^2}{J}\sqrt{E - \mathcal{U}_{\text{eff}}(r)}, \\
\frac{d\theta}{dr} &= \frac{J}{\sqrt{2m}}\frac{1}{r^2\sqrt{E - \mathcal{U}_{\text{eff}}(r)}}, \\
\theta &= \theta_0 + \frac{J}{\sqrt{2m}}\int_{r_0}^{r}\frac{\mathrm{d}r}{r^2\sqrt{E - \frac{1}{2}\frac{J^2}{mr^2} - \frac{1}{2}kr^2}}, \\
&= \theta_0 + J\int_{r_0}^{r}\frac{r^{-1}\mathrm{d}r}{\sqrt{2mEr^2 - J^2 - mkr^4}}, \\
&= \theta_0 + \tfrac{1}{2}J\int_{r_0^2}^{r^2}\frac{\mathrm{d}s}{s\sqrt{2mEs - J^2 - mks^2}},
\end{aligned}$$

where we have used $s = r^2$,

$$ds = 2rdr, \qquad \frac{ds}{2r} = dr, \qquad \frac{ds}{2r^2} = \frac{ds}{2s} = \frac{dr}{r}.$$

Introduce

$$A = J^2, \qquad B = 2mE, \qquad C = mk,$$

and consider the integral

$$\begin{aligned}
\mathcal{I} &= \int\frac{\mathrm{d}s}{s\sqrt{Bs - A - Cs^2}}, \\
&= -\int\frac{\mathrm{d}t}{\sqrt{-At^2 + Bt - C}}, \qquad t = 1/s, \\
&= \frac{1}{\sqrt{A}}\sin^{-1}\left(\frac{-2At + B}{\sqrt{B^2 - 4AC}}\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
\theta-\theta_0 &= \tfrac{1}{2}J\times\left[\frac{1}{J}\sin^{-1}\left(\frac{-(2J^2/r^2)+2mE}{\sqrt{4m^2E^2-4J^2mk}}\right)\right],\\
mE-J^2\frac{1}{r^2} &= \sin 2\left(\theta-\theta_0\right)\sqrt{m^2E^2-J^2mk},\\
\frac{1}{r^2} &= (mE/J^2)-\sin 2\left(\theta-\theta_0\right)\sqrt{(m^2E^2/J^4)-(mk/J^2)},\\
r^2 &= \frac{1}{(mE/J^2)-\sin 2\left(\theta-\theta_0\right)\sqrt{(m^2E^2/J^4)-(mk/J^2)}}.
\end{aligned}$$

Multiply above and below by $J^2/mE$ and obtain

$$r^2=\frac{J^2/mE}{1-\sqrt{1-(J^2k/mE^2)}\sin\left[2\left(\theta-\theta_0\right)\right]}.$$

Identify

$$r_0=\sqrt{J^2/mE},\qquad \delta=\sqrt{1-(J^2k/mE^2)}<1,$$

hence

$$r^2=\frac{r_0^2}{1-\delta\sin\left[2\left(\theta-\theta_0\right)\right]}.$$

Moreover, from the effective potential, $\mathcal{U}_{\text{eff}}'\left(r_0\right)=0\implies r_0=J/\sqrt{mk}$, hence $E\geq\mathcal{U}_{\text{eff}}\left(r_0\right)=J\sqrt{k/m}$. Thus, $0\leq J^2k/mE^2\leq 1$, and the radicand always exists.

Choose $\theta_0$ s.t. $\sin\left[2\left(\theta-\theta_0\right)\right]=\cos 2\theta=\cos^2\theta-\sin^2\theta$. Thus, the orbit equation is

$$r^2-\delta r^2\cos^2\theta+\delta r^2\sin^2\theta=r_0^2,$$

that is,

$$x^2\left(1-\delta\right)+y^2\left(1+\delta\right)=r_0^2.$$

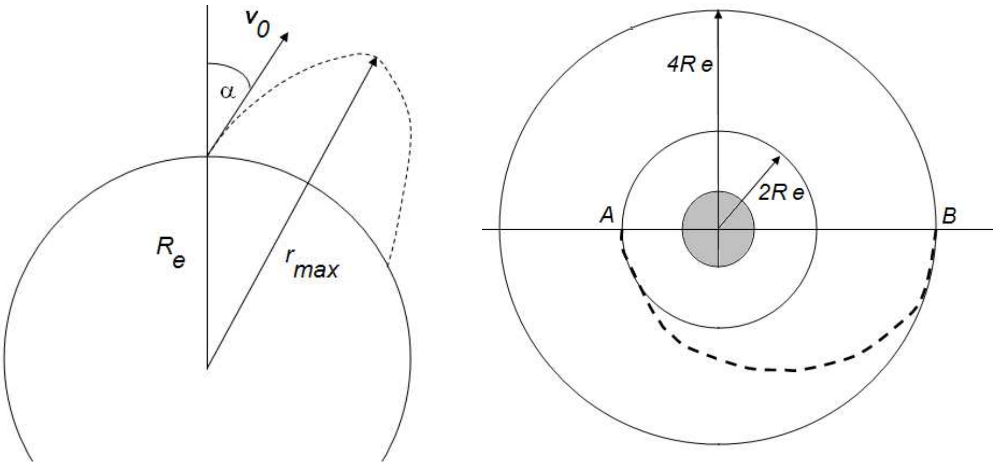
This is the equation of an ellipse with semi-major axis $a=r_0/\sqrt{1-\delta}$ and semi-minor axis $b=r_0/\sqrt{1+\delta}$. The curve is always an ellipse because $\delta<1$. The curve can be re-written as

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

The eccentricity $\epsilon$ is defined by the relation $[\text{minor}]^2=[\text{major}]^2\left(1-\epsilon^2\right)$, hence,

$$\frac{1-\delta}{1+\delta}=1-\epsilon^2,\qquad \epsilon^2=\frac{2\delta}{1+\delta}\leq 1\text{ for }\delta\leq 1.$$

(e) We have shown that the 'solar system on a spring' satisfies Kepler's First Law (it also satisfies Kepler's Second Law – equal areas in equal times). However, it does not satisfy the Thirds Law (periods). It is therefore inconsistent with observations. It is not, however, physically impossible. The effective-potential well is produces bound orbits at all energies, so such a solar system would be highly stable (more stable than the real one). However, with this stability there comes a price: the effective-potential well exerts its pull out to $r = \infty$. Indeed, $r \sim kr^2/2$ as $r \to \infty$. Thus, all matter in the universe would be drawn into the sun's gravitational field. The earth's gravitational field would extend similarly. This would increase the probability of collisions between the earth and other massive bodies, so making life on earth impossible. Thus, the inverse-square law is yet another piece in the jigsaw of design (or coincidence) that makes life on earth possible.




(a) Problem 1 (b) Problem 2

Figure 1: Definition sketches

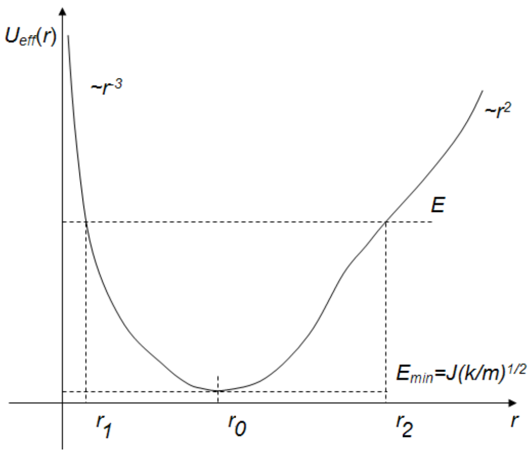



Figure 2: The effective potetnial for $\boldsymbol{F} = -k\boldsymbol{r}$.