Mechanics and Special Relativity (MAPH10030)
Assignment 3 (Bonus marks)

Issue Date: 03 March 2010
Due Date: 24 March 2010

This assignment is optional. If you choose to do it, it carries five marks, which will be added to the score you received in assignment 1, for a maximum of 20 points in assignment 1.

Consider a particle of mass m in two dimensions, experiencing a central force $\boldsymbol{F} = -k\boldsymbol{r}$, where $\boldsymbol{r}$ is the radius vector of the particle relative to the force centre, and in an inertial frame. There are two ways of solving for the motion of such a system. The first way is as to write down the equations of motion in Cartesian form,

$$m\ddot{x} + kx = 0, \qquad m\ddot{y} + ky = 0,$$

and observe that the answer is two uncoupled SHM's, $x = A_x \cos(\omega t + \varphi_x)$, $y = A_y \cos(\omega t + \varphi_y)$, where A_x etc. are arbitrary constants. This solution pair satisfies the generic conic-section equation

$$A\,(x/A_x)^2 + B\,(x/A_x)\,(y/A_y) + C\,(y/A_y)^2 + D\,(x/A_x) + E\,(y/A_y) + F = 0,$$

where

$$\begin{aligned} A &= 1, \\ B &= -2\cos\theta, \qquad \theta = \varphi_x - \varphi_y, \\ C &= 1, \\ D &= 0, \\ E &= 0, \\ F &= -\sin^2\theta. \end{aligned}$$

Hence, $B^2 - 4AC = 4\,(\cos^2\theta - 1) < 0$, and the motion is an ellipse. This is the quick and easy answer. However, the assignment requires that you follow the class notes, and find the answer as follows:

1. What is the angular momentum $\boldsymbol{J}$ for the particle relative to the force centre? Using Newton's laws for a general, N-dimensional one-particle system, show that this is conserved [1 point].

   The angular momentum of the particle about the force centre is

$$\boldsymbol{J} = m\boldsymbol{r} \times \boldsymbol{v}, \qquad \boldsymbol{v} = \dot{\boldsymbol{r}}.$$

Newton's law for the particle is

$$m\ddot{\boldsymbol{r}} = -k\boldsymbol{r}.$$

The rate-of-change of angular momentum is

$$\frac{dJ}{dt} = \boldsymbol{r} \times \frac{d}{dt}(m\boldsymbol{v}) = \boldsymbol{r} \times (-k\boldsymbol{r}) = 0,$$

which we knew already because $\boldsymbol{F}$ is central.

2. Using the polar-coordinate accelerations derived in class, write down the planar version of the equations of motion [1 point].

   Because the angular momentum is conserved, the motion takes place in the plane defined by $\boldsymbol{r}(0)$ and $\boldsymbol{v}(0)$. Therefore we choose a coordinate system (x, y) in this plane, whose origin is at the force centre. The potential for the central force $\boldsymbol{F} = -k\boldsymbol{r}$ is $\mathcal{U} = \frac{1}{2}k|\boldsymbol{r}|^2 = \frac{1}{2}kr^2$, hence $F_r = -\partial\mathcal{U}/\partial r = -kr$, and $F_\theta = -(1/r)(\partial\mathcal{U}/\partial\theta) = 0$. Newton's equations are thus

$$\begin{aligned} m\left(\ddot{r} - r\dot{\theta}^2\right) &= -\frac{\partial\mathcal{U}}{\partial r} = -kr, \\ m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) &= 0. \end{aligned}$$

3. Reduce the system to a one-equation problem and identify the effective potential. Sketch the result [1 point].

   The EOM in the θ direction can be re-written as

$$\frac{m}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0,$$

   hence $J = mr^2\dot{\theta}$ is conserved. This is the magnitude of the angular momentum, because $\boldsymbol{J} = mr\hat{\boldsymbol{r}} \times \left(\dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}\right) = mr^2\dot{\theta}\hat{\boldsymbol{n}}$, where $\hat{\boldsymbol{n}}$ is normal to the plane of motion. Therefore, $J/(mr^2) = \dot{\theta}$, and the centrifugal force in the radial equation becomes

$$mr\dot{\theta}^2 = mr\left(\frac{J}{mr^2}\right)^2 = \frac{J^2}{mr^3}.$$

   The radial equation of motion is

$$\begin{aligned} m\ddot{r} &= \frac{J^2}{mr^3} - kr, \\ &= -\frac{\partial}{\partial r}\left(\frac{1}{2}\frac{J^2}{mr^2} + \frac{1}{2}kr^2\right), \qquad \mathcal{U}_{\text{eff}} = \frac{1}{2}\frac{J^2}{mr^2} + \frac{1}{2}kr^2. \end{aligned}$$

   The effective potential is shown in Fig. 1.

4. Using the class notes as a hint, find the shape of the orbit of the particle. Are the orbits always closed? [1 point]

Hint: Reduce the orbit problem to the integral

$$\theta - \theta_0 = \frac{J}{\sqrt{2m}} \int^r \frac{\mathrm{d}s}{s^2 \sqrt{E - \frac{1}{2}\frac{J^2}{ms^2} - \frac{1}{2}ks^2}},$$

and explain, using Q. 3 why $E > 0$. Then solve for the integral using

$$\mathcal{I} = \int \frac{\mathrm{d}s}{s\sqrt{-Cs^4 + Bs^2 - A}} = \underbrace{-\tfrac{1}{2} \int \frac{\mathrm{d}u}{\sqrt{-Au^2 + Bu - C}}}_{u=1/s^2} =$$

$$\tfrac{1}{2}\frac{1}{\sqrt{A}} \sin^{-1}\left[\frac{-2Au + B}{\sqrt{B^2 - 4AC}}\right],$$

where A, B, and C are positive constants.

From the effective potential, obtain the conserved energy

$$E = \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}\frac{J^2}{mr^2} + \tfrac{1}{2}kr^2 = \tfrac{1}{2}m\dot{r}^2 + \mathcal{U}_{\text{eff}}(r).$$

The energy for this potential well is never negative.

The equation for dr/dt is found from the conserved energy:

$$\frac{dr}{dt} = \sqrt{\frac{2}{m}}\sqrt{E - \mathcal{U}_{\text{eff}}(r)}.$$

We also know $d\theta/dt$ from angular-momentum conservation:

$$\frac{d\theta}{dt} = \frac{J}{mr^2}.$$

Dividing these equations one by the other and using the chain rule gives $dr/d\theta$:

$$\begin{aligned}
\frac{dr}{d\theta} &= \sqrt{\frac{2}{m}}\frac{mr^2}{J}\sqrt{E - \mathcal{U}_{\text{eff}}(r)}, \\
\frac{d\theta}{dr} &= \frac{J}{\sqrt{2m}}\frac{1}{r^2\sqrt{E - \mathcal{U}_{\text{eff}}(r)}}, \\
\theta &= \theta_0 + \frac{J}{\sqrt{2m}}\int_{r_0}^{r} \frac{\mathrm{d}r}{r^2\sqrt{E - \frac{1}{2}\frac{J^2}{mr^2} - \frac{1}{2}kr^2}}, \\
&= \theta_0 + J\int_{r_0}^{r} \frac{r^{-1}\mathrm{d}r}{\sqrt{2mEr^2 - J^2 - mkr^4}}, \\
&= \theta_0 + \tfrac{1}{2}J\int_{r_0^2}^{r^2} \frac{\mathrm{d}s}{s\sqrt{2mEs - J^2 - mks^2}},
\end{aligned}$$

where we have used $s = r^2$,

$$ds = 2rdr, \qquad \frac{ds}{2r} = dr, \qquad \frac{ds}{2r^2} = \frac{ds}{2s} = \frac{dr}{r}.$$

Use

$$A = J^2, \qquad B = 2mE, \qquad C = mk$$

Hence,

$$r^2 = \frac{J^2/mE}{1 - \sqrt{1 - (J^2 k/mE^2)} \sin\left[2\left(\theta - \theta_0\right)\right]}.$$

Identify

$$r_0 = \sqrt{J^2/mE}, \qquad \delta = \sqrt{1 - (J^2 k/mE^2)} < 1,$$

hence

$$r^2 = \frac{r_0^2}{1 - \delta \sin\left[2\left(\theta - \theta_0\right)\right]}.$$

Moreover, from the effective potential, $\mathcal{U}'_{\text{eff}}\left(r_0\right) = 0 \implies r_0 = J/\sqrt{mk}$, hence $E \geq \mathcal{U}_{\text{eff}}\left(r_0\right) = J\sqrt{k/m}$. Thus, $0 \leq J^2 k/mE^2 \leq 1$, and the radicand always exists.

Choose θ_0 s.t. $\sin\left[2\left(\theta - \theta_0\right)\right] = \cos 2\theta = \cos^2\theta - \sin^2\theta$. Thus, the orbit equation is

$$r^2 - \delta r^2 \cos^2\theta + \delta r^2 \sin^2\theta = r_0^2,$$

that is,

$$x^2\left(1 - \delta\right) + y^2\left(1 + \delta\right) = r_0^2.$$

This is the equation of an ellipse with semi-major axis $a = r_0/\sqrt{1-\delta}$ and semi-minor axis $b = r_0/\sqrt{1+\delta}$. The curve is always an ellipse because $\delta < 1$. The curve can be re-written as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The eccentricity ϵ is defined by the relation $[\text{minor}]^2 = [\text{major}]^2\left(1 - \epsilon^2\right)$, hence,

$$\frac{1-\delta}{1+\delta} = 1 - \epsilon^2, \qquad \epsilon^2 = \frac{2\delta}{1+\delta} \leq 1 \text{ for } \delta \leq 1.$$

5. If the orbit is closed, show that the area swept out by the radius vector is constant.

   From the class notes, all central forces conserve angular momentum, and therefore sweep out equal areas in equal times. In symbols,

$$dA = \tfrac{1}{2} r^2 d\theta = \tfrac{1}{2} r^2 \dot{\theta} dt = \frac{J}{2m} dt,$$

   hence

$$\frac{dA}{dt} = \frac{J}{m}.$$

6. Would a solar system governed by such a force law make sense? [0 points, but please think about this. The answer makes life on earth possible.]

   We have shown that the 'solar system on a spring' satisfies Kepler's first and second laws. However, it does not satisfy the law of periods. It is therefore inconsistent with observations. It is not, however, physically impossible. The effective-potential well is produces bound orbits at all energies, so such a solar system would be highly stable (more stable than the real one). However, with this stability there comes a price: the effective-potential well exerts its pull out to $r = \infty$. Indeed, $r \sim kr^2/2$ as $r \to \infty$. Thus, all matter in the universe would be drawn into the sun's gravitational field. The earth's gravitational field would extend similarly. This would increase the probability of collisions between the earth and other massive bodies, so making life on earth impossible. Thus, the inverse-square law is yet another piece in the jigsaw of coincidences (or design) that makes life on earth possible.

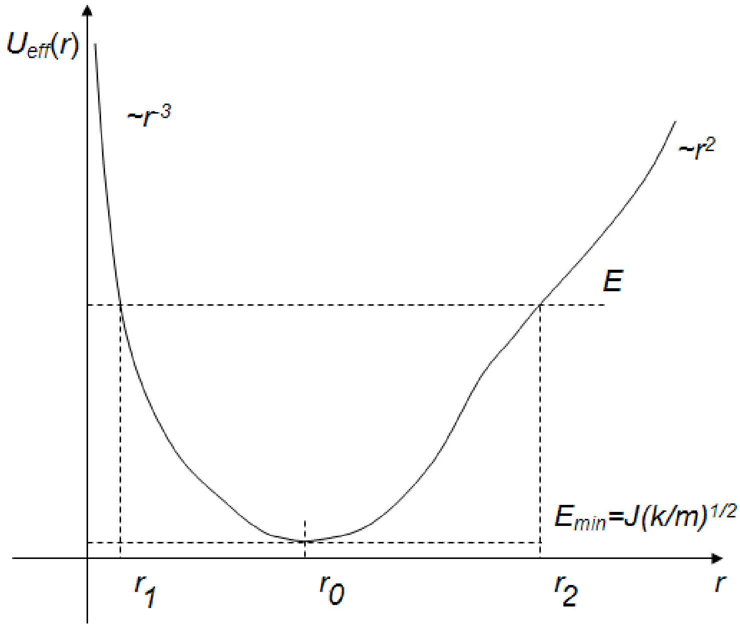

Figure 1: The effective potetnial for $\boldsymbol{F} = -k\boldsymbol{r}$.