Mechanics and Special Relativity (ACM10030) Assignment 2

Issue Date: 28th February 2011
Due Date: 21st March 2011

1. **Force and torque** One force acting on a machine part is $\boldsymbol{F} = (-5.00\,\mathrm{N})\,\hat{\boldsymbol{x}} + (4.00\,\mathrm{N})\,\hat{\boldsymbol{y}}$. The vector from the origin to the point where the force is applied is $\boldsymbol{r} = (-0.450\,\mathrm{m})\,\hat{\boldsymbol{x}} + (0.150\,\mathrm{m})\,\hat{\boldsymbol{y}}$.

   (a) In a sketch show $\boldsymbol{r}$, $\boldsymbol{F}$, and the origin. You must show $\boldsymbol{r}$ and $\boldsymbol{F}$ on different sets of axes because they have different physical units.

   (b) Use the right-hand rule to determine the direction of the torque. Then, compute the torque from the determinant definition. Make sure that the direction obtained in both calculations is the same.

2. **Angular momentum and torque**

   (a) Show that if the total linear momentum of a system of particles is zero, the angular momentum of the system is the same about all origins.

   (b) Show that if the total force on a system of particles is zero, the torque on the system is the same about all origins.

3. **Gravitational forces on extended bodies** Recall the law of gravity for point particles m_1 and m_2: the force on particle 1 due to particle 2 is given by

$$\boldsymbol{F}_{12} = -\frac{Gm_1m_2}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|^2}\left(\frac{\boldsymbol{x}_1 - \boldsymbol{x}_2}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|}\right). \tag{1}$$

   In class, we stated that the same law holds for spherical bodies at finite separations, and that the proof of this statement follows by integration. In this problem we obtain a hint at how this integration might be done by considering the gravitational force exerted by a continuous line of particles on a point particle of mass m.

   Consider the system shown in Fig. 1(a). A continuous line of particles extends from $x = -a$ to $x = a$, at $y = 0$. A point mass lies at $x = 0$, $y = L$.

   (a) Show that the force on the particle due to a point-like mass $\mathrm{d}m\,(x)$ extending from x to $x + \mathrm{d}x$ is

$$\mathrm{d}\boldsymbol{F}_{1,x} = -\frac{Gm\,\mathrm{d}m\,(x)}{(x^2 + L^2)^{3/2}}\left(L\hat{\boldsymbol{y}} - x\hat{\boldsymbol{x}}\right).$$

(b) Assume a linear mass density $\mathrm{d}m = \rho\,\mathrm{d}x$ ($\rho =$ Const.) and thus obtain the total force $\boldsymbol{F}_1$ on the point mass m. Use a table of integrals if necessary.

4. **Gravitational self-energy** Consider a solid sphere of uniform density ρ, radius R_0, and mass M.

   (a) Explain why the gravitational interaction between a mass element $\mathrm{d}m$ and a solid sphere of radius r and constant density ρ – where the mass element sits on the surface of the sphere – is given by

   $$\mathrm{d}\mathcal{U} = -G\mathrm{d}m\left(\tfrac{4}{3}\pi r^3\rho/r\right).$$

   (b) By integrating over all such mass elements that sit in a shell of thickness $\mathrm{d}r$ on the surface of the sphere in part (a), show that the gravitational interaction between the shell and the sphere is

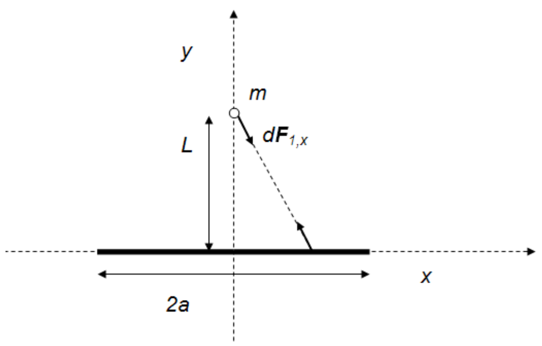
   $$\mathrm{d}\mathcal{U} = -\tfrac{16}{3}\pi^2 G\rho^2 r^4\,\mathrm{d}r.$$

   (c) Do one final integration to show that the gravitational self-energy of the sphere is

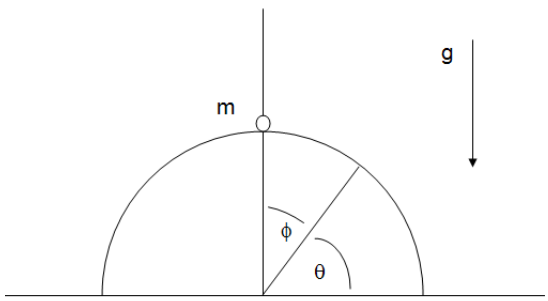
   $$\mathcal{U} = -\tfrac{3}{5}\frac{GM^2}{R_0}.$$

5. **Bonus problem: This question is not mandatory, but can be used to top up the marks on the other questions, for a maximum of five top-up marks.** Consider a particle that is constrained on top of a semicircle (Fig. 1(b)). Gravity points downwards. Suppose that the particle starts from rest. At what angle does the particle fall off the semicircle?

   Give the solution in two forms: in terms of the angle ϕ, and the angle θ.



(a) Gravitational interaction between a particle and a rod.



(b) Sketch for bonus problem.

Figure 1: