Mechanics and Special Relativity (ACM10030)
Assignment 1

Issue Date: 02 February 2010
Due Date: 09 February 2010

Each question carries five marks.

1. Recall the equations of trajectory motion in a uniform gravitational field g:

$$x = x_0 + u_0 t, \tag{1a}$$
$$y = y_0 + v_0 t - \tfrac{1}{2} g t^2. \tag{1b}$$

   where (x_0, y_0) is the initial location of the particle relative to a given inertial frame and $\boldsymbol{v}_0 := (u_0, v_0)$ is the initial velocity. Neglect air resistance.

   A girl throws a water balloon at an angle α above the horizontal with a speed $|\boldsymbol{v}_0|$. The horizontal component of the balloon's velocity is directed towards a car that is approaching the girl with a constant speed V. If the balloon is to hit the car at the same height at which it leaves her hand, what is the maximum distance the car can be from the girl when the balloon is thrown?

   The answer, H, involves V, $|\boldsymbol{v}_0|$, α, and g.

2. Consider a particle experiencing the force $F = +kx$, a repulsive spring force (note the POSITIVE sign!!).

   (a) Write down the equation of motion and the energy.

   (b) Reduce the motion to an integral using the energy. Focus on the case where the energy is positive.

   (c) Solve this integral and find $x(t)$.

   Hint:

$$\int \frac{\mathrm{d}y}{\sqrt{1+y^2}} = \sinh^{-1}(y) + \text{Const.}, \qquad \sinh y = \frac{e^y - e^{-y}}{2}.$$

3. Consider the potential

$$\mathcal{U}(x) = \tfrac{1}{2} m\omega^2 x^2 - \tfrac{1}{4} m\lambda^2 x^4,$$

   where ω and λ are positive constants, and where m is the particle mass. Find the points of unstable equilibrium, the point of stable equilibrium, and the period of small oscillations about the stable equilibrium. Sketch the potential function and mark in the equilibrium points.

4. Consider a particle moving about the bottom of a potential well. We know from class that

$$E = \tfrac{1}{2}m\dot{x}^2 + \mathcal{U}\left(x\right),$$

and hence that

$$\frac{dx}{dt} = \sqrt{\frac{2}{m}}\sqrt{[E-\mathcal{U}\left(x\right)]}, \qquad \frac{dt}{dx} = \sqrt{\frac{m}{2}}\frac{1}{\sqrt{[E-\mathcal{U}\left(x\right)]}}.$$

The **turning-points** x_1 and x_2 of the motion occur at $dx/dt = 0$, or $E = \mathcal{U}(x)$, and the **half-period** is the time required by the particle to go from one turning-point to another (See Fig. 1).

$$\tfrac{1}{2}T = \sqrt{\frac{m}{2}}\int_{x_1}^{x_2}\frac{\mathrm{d}x}{\sqrt{[E-\mathcal{U}\left(x\right)]}}.$$

Now, consider a spring that exerts the following **quartic** restoring potential:

$$\mathcal{U}(x) = \tfrac{1}{4}m\lambda^2 x^4$$

(a) If the particle has mass m and is released from rest at $x = A$, prove that the half-period can be written as

$$\tfrac{1}{2}T = [\text{Some function of } m, E, \text{ and } \lambda] \\ \times [\text{Some integral independent of the mechanical parameters}]$$

It is required that you derive these functions explicitly.

(b) Does the period depend on A? Would the period depend on A if $\mathcal{U}(x)$ were a quadratic potential?

Hint: You may need the following substitution:

$$y = \left(\tfrac{1}{4}\frac{m\lambda^2}{E}\right)^{1/4} x.$$

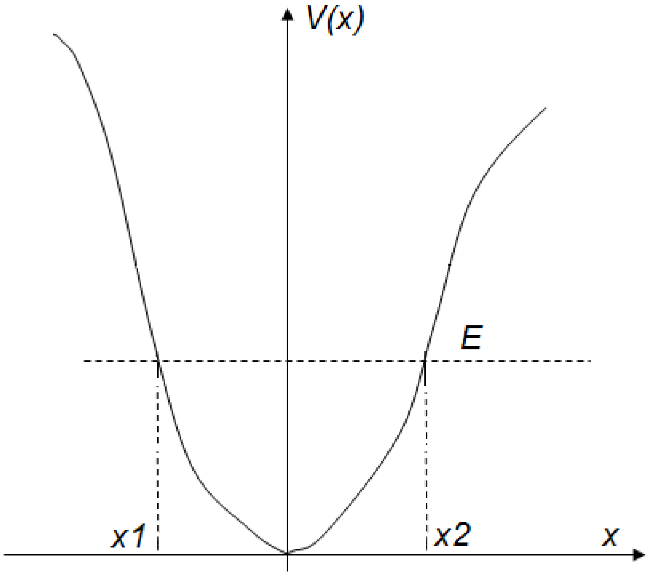

Figure 1: The turning points x_1 and x_2 for a typical potential well.