Mechanics and Special Relativity (ACM10030) Assignment 1

Issue Date: 02 February 2010
Due Date: 09 February 2010

1. Recall the equations of trajectory motion in a uniform gravitational field g:

$$x = x_0 + u_0 t, \tag{1a}$$
$$y = y_0 + v_0 t - \tfrac{1}{2} g t^2. \tag{1b}$$

where (x_0, y_0) is the initial location of the particle relative to a given inertial frame and $\boldsymbol{v}_0 := (u_0, v_0)$ is the initial velocity. Neglect air resistance.

A girl throws a water balloon at an angle α above the horizontal with a speed $|\boldsymbol{v}_0|$. The horizontal component of the balloon's velocity is directed towards a car that is approaching the girl with a constant speed V. If the balloon is to hit the car at the same height at which it leaves her hand, what is the maximum distance the car can be from the girl when the balloon is thrown?

The answer, H, involves V, $|\boldsymbol{v}_0|$, α, and g.

We are to consider the foremost tip of the car. We ask the question, at what time does the balloon hit the tip, assuming that the collision occurs at the launch height. The initial velocity of the balloon is $\boldsymbol{v}_0 = (u_0, v_0) = |\boldsymbol{v}_0| (\cos\alpha, \sin\alpha)$. We work in the FOR of the earth with a choice of origin $(x_0, y_0) = (0, 0)$. Hence,

$$x = |\boldsymbol{v}_0| \cos\alpha\, t, \qquad y = |\boldsymbol{v}_0| \sin\alpha\, t - \tfrac{1}{2} g t^2.$$

The coordinate of the car in this frame is

$$x_{\mathrm{car}} = H - Vt,$$

where the minus sign indicates that the car is approaching the girl, who is fixed to the FOR of the earth. To find the time of collision, form the following equality:

$$x = x_{\mathrm{car}} \implies |\boldsymbol{v}_0| \cos\alpha\, t = H - Vt \implies (|\boldsymbol{v}_0| \cos\alpha + V)\, t = H.$$

Hence,

$$t_{\mathrm{coll}} = \frac{H}{|\boldsymbol{v}_0| \cos\alpha + V}.$$

Rather unsurprisingly, the collision is hastened by the car's having a finite velocity in the girl's direction.

Now we find H. At the collision time, $y = y_{\rm car}$ too. This location is at $y = 0$. Hence, $|\boldsymbol{v}_0| \sin\alpha t_{\rm c} - \frac{1}{2} g t_{\rm c}^2 = 0$. Assuming $t_{\rm c} \neq 0$, obtain

$$t_{\rm c} = \frac{2 v_0 \sin\alpha}{g}.$$

Now we have two equations for $t_{\rm c}$. We equate them and solve for H,

$$t_{\rm c} = \frac{2|\boldsymbol{v}_0| \sin\alpha}{g} = \frac{H}{v_0 \cos\alpha + V},$$

hence

$$H = 2|\boldsymbol{v}_0| \sin\alpha \,(V + |\boldsymbol{v}_0| \cos\alpha)\, g^{-1}.$$

2. Consider a particle experiencing the force $F = +kx$, a repulsive spring force (note the POSITIVE sign!!).
   (a) Write down the equation of motion and the energy.
   (b) Reduce the motion to an integral using the energy. Focus on the case where the energy is positive.
   (c) Solve this integral and find $x(t)$.

   Hint:

$$\int \frac{\mathrm{d}y}{\sqrt{1+y^2}} = \sinh^{-1}(y) + \text{Const.}, \qquad \sinh y = \frac{e^y - e^{-y}}{2}.$$

We have the force $F = +kx$, hence the potential is $\mathcal{U} = -kx^2/2$, and the conserved energy is

$$E = \tfrac{1}{2} m \left(\frac{dx}{dt}\right)^2 - \tfrac{1}{2} k x^2,$$

Calling $\sigma = \sqrt{k/m}$, this is

$$E = \tfrac{1}{2} m \left(\frac{dx}{dt}\right)^2 - \tfrac{1}{2} m \sigma^2 x^2.$$

The energy is not positive definite. However, we focus on the case where the initial conditions conspire to give $E > 0$. Inverting for dt/dy,

$$\begin{aligned}
\frac{dx}{dt} &= \sqrt{\frac{2E}{m}} \sqrt{1 + \frac{1}{2}\frac{m\sigma^2 x^2}{E}}, \\
\frac{dt}{dx} &= \sqrt{\frac{m}{2E}} \frac{1}{\sqrt{1 + \frac{1}{2}\frac{m\sigma^2 x^2}{E}}}, \\
t &= \sqrt{\frac{m}{2E}} \int_{x_0}^{x} \frac{\mathrm{d}x}{\sqrt{1 + \frac{1}{2}\frac{m\sigma^2 x^2}{E}}}.
\end{aligned}$$

We transform to dimensionless variables: $y^2 = m\sigma^2 x^2/(2E)$. The integral is thus

$$t = \frac{1}{\sigma}\int_{y_0}^{y}\frac{\mathrm{d}y}{\sqrt{1+y^2}},$$

where $y_0 = x_0(m\sigma^2/2E)^{1/2}$ and $y = x(m\sigma^2/2E)^{1/2}$. The integral is given:

$$t = \frac{1}{\sigma}\left[\sinh^{-1} y - \sinh^{-1} y_0\right].$$

Define a constant of integration $\tilde{A}$,

$$\tilde{A} = \sinh^{-1} y_0$$

Hence,

$$\sigma t = \sinh^{-1} y - \tilde{A} \iff y = \sinh\left(\sigma t + \tilde{A}\right),$$

and, restoring the x-coordinate, this is

$$x = \sqrt{\frac{2E}{m\sigma^2}}\sinh\left(\sigma t + \tilde{A}\right).$$

Defining a further constant of integration

$$\tilde{B} = \sqrt{\frac{2E}{m\sigma^2}},$$

the solution is

$$x = \tilde{B}\sinh\left(\sigma t + \tilde{A}\right).$$

Using $\sinh s = \left(e^s - e^{-s}\right)/2$, this is

$$x = \tfrac{1}{2}\tilde{B}e^{\tilde{A}}e^{\sigma t} - \tfrac{1}{2}\tilde{B}e^{-\tilde{A}}e^{-\sigma t}.$$

Defining further constants of integration $A = \tilde{B}e^{\tilde{A}}/2$ and $B = -\tilde{B}e^{-\tilde{A}}/2$, this is

$$x = Ae^{\sigma t} + Be^{-\sigma t},$$

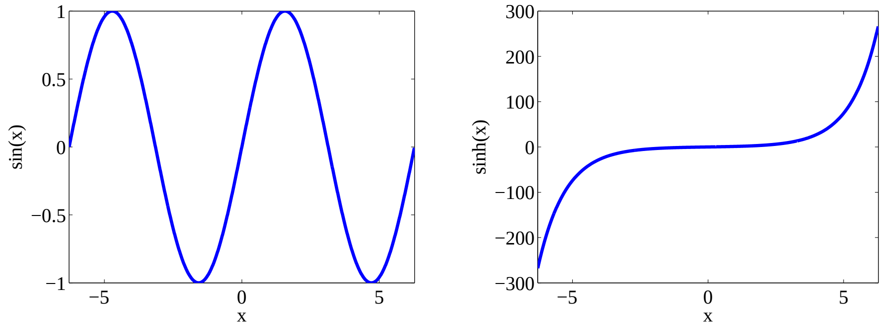
and A and B can be fixed by the initial conditions on y and $\dot{y}$.

Note: The functions $\sin$ and $\sinh$ could not be more different. The function $\sin$ is periodic; the function $\sinh$ blows up exponentially as its argument tends to $\pm\infty$ (See Fig. 1).

3. Consider the potential

$$\mathcal{U}(x) = \tfrac{1}{2}m\omega^2 x^2 - \tfrac{1}{4}m\lambda^2 x^4,$$

where ω and λ are positive constants, and where m is the particle mass. Find the points of unstable equilibrium, the point of stable equilibrium, and the period of small oscillations about the stable equilibrium. Sketch the potential function and mark in the equilibrium points.



(a) (b)

Figure 1: The difference between $\sin$ and $\sinh$.

From class notes, equilibrium corresponds to $\mathcal{U}'(x) = 0$. Now,

$$\mathcal{U}'(x) = m\omega^2 x - m\lambda^2 x^3.$$

Setting this to zero gives $x = 0$ or

$$\omega^2 = \lambda^2 x^2.$$

Hence, the equilibria are

$$\begin{aligned} x_0 &= 0, \\ x_{\pm\lambda} &= \pm\frac{\omega}{\lambda}. \end{aligned}$$

The stability or otherwise of the equilibria is characterised by the second derivative of $\mathcal{U}(x)$:

$$\mathcal{U}''(x) = m\omega^2 - 3m\lambda^2 x^2.$$

We have,

$$\begin{aligned} \mathcal{U}''(x_0) &= m\omega^2 > 0, \quad \cdots \ \text{stable}, \\ \mathcal{U}''(x_{\pm\lambda}) &= m\omega^2 - 3m\lambda^2\left(\frac{\omega^2}{\lambda^2}\right) = -2m\omega^2 < 0, \quad \cdots \ \text{unstable}. \end{aligned}$$

The frequency of small oscillations around $x_0 = 0$ is

$$\sqrt{\mathcal{U}''(x_0)/m} = \sqrt{m\omega^2/m} = \omega.$$

The period is therefore

$$T = 2\pi/\omega.$$

A sketch of the potential well is shown in Fig. 2.

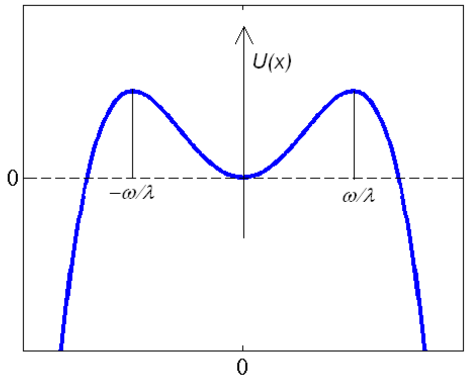


Figure 2: Quadratic-quartic potential well

4. Consider a particle moving about the bottom of a potential well. We know from class that

$$E = \tfrac{1}{2}m\dot{x}^2 + \mathcal{U}(x),$$

and hence that

$$\frac{dx}{dt} = \sqrt{\frac{2}{m}}\sqrt{[E-\mathcal{U}(x)]}, \qquad \frac{dt}{dx} = \sqrt{\frac{m}{2}}\frac{1}{\sqrt{[E-\mathcal{U}(x)]}}.$$

The **turning-points** x_1 and x_2 of the motion occur at $dx/dt = 0$, or $E = \mathcal{U}(x)$, and the **half-period** is the time required by the particle to go from one turning-point to another (See Fig. 3).

$$\tfrac{1}{2}T = \sqrt{\frac{m}{2}}\int_{x_1}^{x_2}\frac{\mathrm{d}x}{\sqrt{[E-\mathcal{U}(x)]}}.$$

Now, consider a spring that exerts the following **quartic** restoring potential:

$$\mathcal{U}(x) = \tfrac{1}{4}m\lambda^2 x^4$$

(a) If the particle has mass m and is released from rest at $x = A$, prove that the half-period can be written as

$$\begin{aligned}\tfrac{1}{2}T = {} & [\text{Some function of } m, E, \text{ and } \lambda] \\ & \times [\text{Some integral independent of the mechanical parameters}]\end{aligned}$$

It is required that you derive these functions explicitly.

(b) Does the period depend on A? Would the period depend on A if $\mathcal{U}(x)$ were a quadratic potential?

Hint: You may need the following substitution:

$$y = \left(\tfrac{1}{4}\frac{m\lambda^2}{E}\right)^{1/4} x.$$

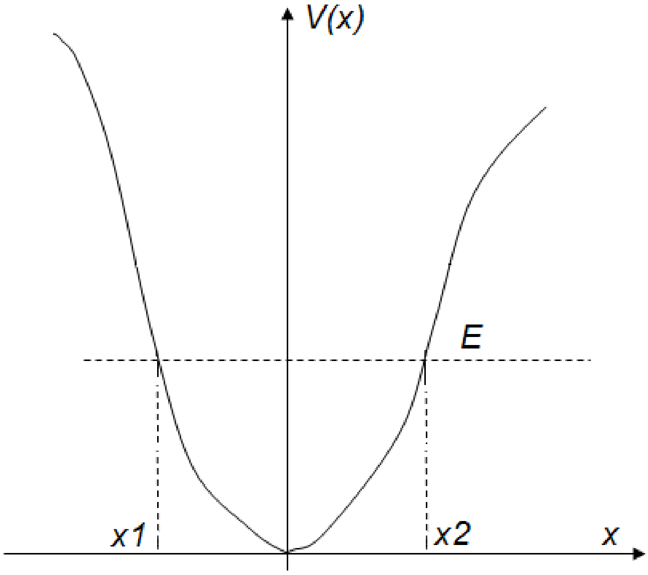


Figure 3: The turning points x_1 and x_2 for a typical potential well.

We have

$$E = E(t = 0) = \text{no kinetic energy} + \tfrac{1}{4} m\lambda^2 A^4.$$

Thus, a general expression for the energy is

$$\tfrac{1}{4} m\lambda^2 A^4 = \tfrac{1}{2} m\dot{x}^2 + \tfrac{1}{4} m\lambda^2 x^4,$$

and

$$\tfrac{1}{2} m\dot{x}^2 = \tfrac{1}{4} m\lambda^2 \left(A^4 - x^4\right) = E - \mathcal{U}(x).$$

The turning points of the motion (where $\dot{x}$ vanishes) are therefore $x_1 = -A$ and $x_2 = +A$. We now have enough material to write down the half-period:

$$\tfrac{1}{2} T = \sqrt{\frac{m}{2}} \int_{-A}^{A} \frac{\mathrm{d}x}{\sqrt{E - \frac{1}{4} m\lambda^2 x^4}}.$$

As usual, we take a factor of E outside downstairs in the square root. There is no ambiguity here since E is necessarily positive.

$$\tfrac{1}{2} T = \sqrt{\frac{m}{2E}} \int_{-A}^{A} \frac{\mathrm{d}x}{\sqrt{1 - \frac{1}{4}\frac{m\lambda^2 x^4}{E}}}.$$

Let's use the substitution:

$$y = \left(\frac{m\lambda^2}{4E}\right)^{1/4} x.$$

The upper limit is

$$\begin{aligned} y_{\text{upper}} &= \left(\frac{m\lambda^2}{4E}\right)^{1/4} A, \\ &= \left(\frac{m\lambda^2 A^4}{4E}\right)^{1/4}, \qquad E = \tfrac{1}{4} m\lambda^2 A^4, \\ &= 1. \end{aligned}$$

Similarly, $y_{\rm lower} = -1$. Note also,

$$\mathrm{d}x = \left(\frac{4E}{m\lambda^2}\right)^{1/4} \mathrm{d}y.$$

Putting it all together,

$$\tfrac{1}{2}T = \sqrt{\frac{m}{2E}} \left(\frac{4E}{m\lambda^2}\right)^{1/4} \int_{-1}^{1} \frac{\mathrm{d}y}{\sqrt{1-y^4}}. \qquad (*)$$

This is the final answer. The integral is a pure number that is independent of the mechanical properties like energy, mass, and the potential constant λ.

For the second part, let us elaborate on the answer. Calling the pure integral I, we have

$$\begin{aligned} \tfrac{1}{2}T &= I\left(\frac{m^2}{4E^2}\frac{4E}{m\lambda^2}\right)^{1/4}, \\ &= I\left(\frac{m}{\lambda^2 E}\right)^{1/4}, \\ &= I\left(\frac{m}{\lambda^2\left(\frac{1}{4}m\lambda^2 A^4\right)}\right)^{1/4}, \\ &= I\left(\frac{4}{\lambda^4 A^4}\right)^{1/4}. \end{aligned}$$

To four significant figures, the integral I is calculated numerically as $I = 2.622$. Finally,

$$T = \frac{2\sqrt{2}I}{\lambda A}.$$

This makes sense dimensionally: The dimensions of λ are

$$[\lambda] = \frac{1}{\text{Length} \times \text{Time}},$$

hence

$$\left[\frac{1}{\lambda A}\right] = \frac{1}{\frac{1}{\text{Length}\times\text{Time}} \times \text{Length}} = \text{Time},$$

which is the correct dimension for the period.

As for the harmonic oscillator, the period can be obtained from class notes, OR from replacing powers of 4 with powers of 2 in the expression (*). (To be utterly consistent, we should also replace $\lambda^2/4$ with $\omega^2/2$ in the potential function.) The result of the latter procedure is

$$\tfrac{1}{2}T = \sqrt{\frac{m}{2E}} \left(\frac{2E}{m\omega^2}\right)^{1/2} \int_{-1}^{1} \frac{\mathrm{d}y}{\sqrt{1-y^2}}.$$

But now the factors involving energy cancel exactly!!

$$\tfrac{1}{2}T = \frac{1}{\omega}\int_{-1}^{1}\frac{\mathrm{d}y}{\sqrt{1-y^2}}.$$

The integral is a known one in this case: it is equal to π. Thus,

$$\tfrac{1}{2}T = \pi/\omega \implies T = 2\pi/\omega.$$

This is totally independent of the amplitude A of the oscillation. You should note that the cancellation of the energy in the expression

$$\sqrt{\frac{m}{2E}}\left(\frac{pE}{m\lambda^2}\right)^{1/p}$$

relies on p being equal to 2. Thus, only for quadratic potentials is the oscillation period independent of amplitude.