In this lecture, we look at numerical solutions of the SIR model. We also look at some final concepts in mathematical epidemiology:

- Late-stage burnout
- Herd-Immunity Threshold

We start by looking at the numerical solutions. Indeed, we have to take a step further back and look at:

3.4 The three-equation model revisited

Here, we try first of all to make sense of the fixed point u_*, which solves $f(u_*) = 0$ in the reduced SIR model $\left(du/d\tau = f(u) \right)$.

Since u_* is the equilibrium solution, it must also be the equilibrium solution of the full three-equation model, with $dS/dt = dI/dt = dR/dt = 0$. We take $R_* = (\gamma u_* / \beta) N$ (scaling) and we look at the corresponding equilibrium value S_*. We must have:

$$\frac{dS}{dt} = 0 = -\frac{\beta}{N} I_* \cdot S_*$$

$$\frac{dI}{dt} = 0 = \frac{\beta}{N} I_* S_* - \gamma I_*$$

$$\frac{dR}{dt} = 0 = \gamma I_*.$$

dt

So clearly the eq^{m} is :

$$\boxed{R_* = \left(\frac{\gamma u_*}{\beta}\right) N, \qquad I_* = 0, \qquad S_* = N - R_*} \quad (1)$$

We now show further how to construct a general solution ($S(t)$, $I(t)$, $R(t)$) from $u(t)$.

1. S/S_0 : From previous calculations, this is :

$$S/S_0 = e^{-u}.$$

2. R/S_0. We have $R = \left(\frac{\gamma}{\beta}\right) . N . u$,

$$\Rightarrow \frac{R}{S_0} = \left(\frac{\gamma}{\beta} \frac{N}{S_0}\right) u$$

$$\Rightarrow R/S_0 = \left(1/R_0 \right) u.$$

3. I/S_0. By conservation, we have $S + I + R = N$, hence :

a.

$$\frac{I}{S_0} = \frac{N}{S_0} - \frac{S}{S_0} - \frac{R}{S_0}$$

$$\Rightarrow \frac{I}{\ } = a - e^{-u} - \frac{1}{\ } u.$$

$$\Longrightarrow \quad \frac{I}{S_0} = a - e^{-u} - \frac{1}{R_0} u .$$

So in summary, we can reconstruct a (scaled) three-equation SIR model as follows:

$$\left\{ \begin{aligned} &\frac{du}{d\tau} = f(u), \quad f(u) = a - e^{-u} - \frac{1}{R_0} u, \quad \tau > 0, \\ &u(0) = 0 . \end{aligned} \right\} \quad (2a)$$

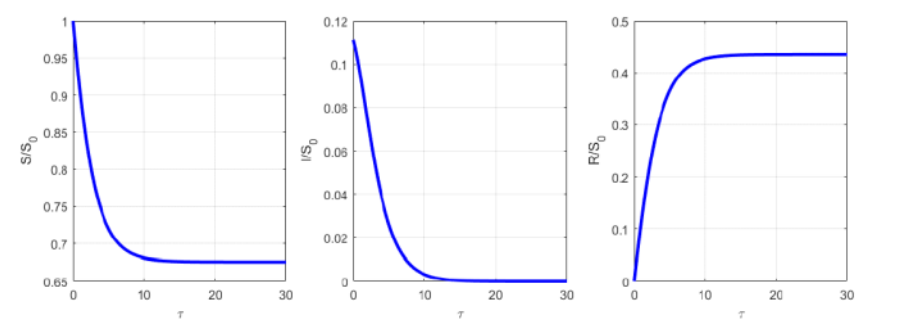
$$\left\{ \begin{aligned} S/S_0 &= e^{-u} \\ I/S_0 &= a - \frac{1}{R_0} u - e^{-u} \\ R/S_0 &= \frac{1}{R_0} u \end{aligned} \right\} \quad (2b)$$

These equations can be put into an ODE solver (e.g. Matlab):

```matlab
function [S,I,R,t]=sir_solve_u(a,R0,Tfin)

options = odeset('RelTol',1e-8,'AbsTol',1e-8,'Stats','on','OutputFcn',@odeplot);
[T,u]=ode45(@myfun,[0 , Tfin],0,options);

S=exp(-u);
I=a-(1/R0)*u-exp(-u);
R=(1/R0)*u;
t=T;

function dudt=myfun(~,u)
    dudt=a-(1/R0)*u-exp(-u);
end

end
```

Results below. Both cases: $a = 10/9$, so $S_0/N = 0.9$, so $I_0/N = 0.1$, hence 10% of the population is initially infected.

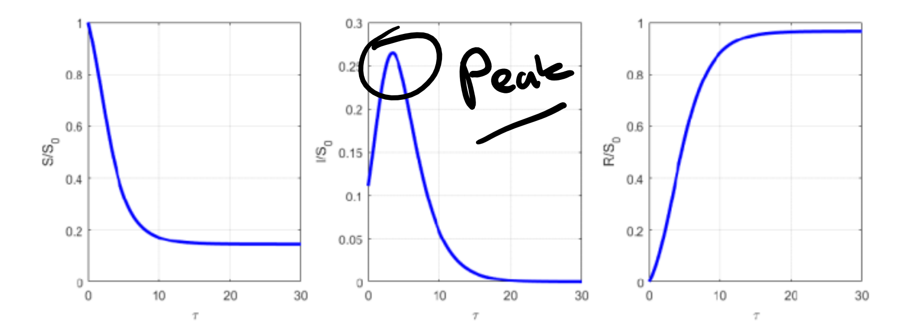
Case #1: $R_0 = 0.9$



Numerical solution of the SIR model, $a = 10/9$, $R_0 = 0.9$

Outbreak dies out from get-go.

Case #2: $R_0 = 2$.



Numerical solution of the SIR model, $a = 10/9$, $R_0 = 2$

Epidemic: Outbreak peaks before dying out.

Other numerical method: Solve full 3-eq^{n} model directly. Model code shown in lecture notes. This will be the approach used in the main projects.

Some final key concepts

Late-stage burnout: From the figures, we see that

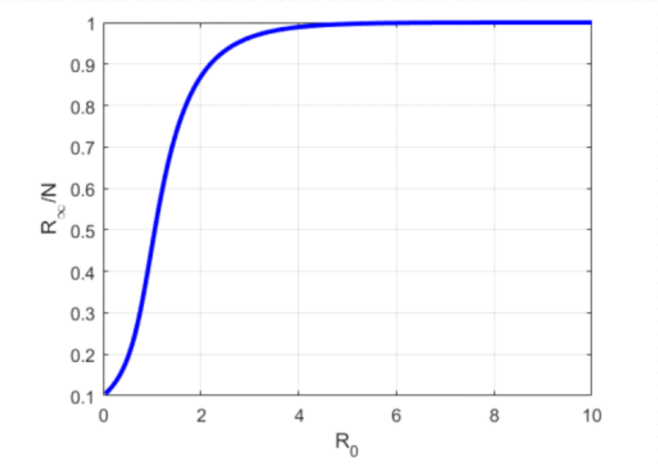
$$\lim_{t \to \infty} R(t) = R_\infty < N.$$

Not everyone gets infected by the disease. The fact that $R_\infty \neq N$ is called "late stage burnout".

fact that $R_\infty \not\Rightarrow N$ is called "late stage burnout".

We can calculate R_∞ NUMERICALLY as a f^{n} of R_0. The higher the value of R_0, the closer R_∞ is to N.

See the figure below.



Plot showing the idea behind late-stage burnout

Herd-Immunity Threshold:

Suppose there is a vaccine for the disease. We vaccinate the susceptible population. To prevent an epidemic, it suffices to vaccinate a fraction $f < 1$ of the susceptible population. To work out what f is, consider the following.

After vaccination of a population with no prior exposure to the disease we have, at $t = 0$:

$I_0 = 1$ ... patient zero

$$S_0 = N - 1 - fN$$

Patient zero   Fraction of population vaccinated

Patient zero   Fraction of population vaccinated

The value of R_0 for the outbreak is now reduced to R_{eff}. We have:

- No vaccination:

$$R_0 = \frac{\beta}{\gamma}\frac{S_0}{N} = \frac{\beta}{\gamma}\frac{N-1}{N} \simeq \frac{\beta}{\gamma}.$$

- With vaccination:

$$R_{eff} = \frac{\beta}{\gamma}\frac{S_0}{N} = \frac{\beta}{\gamma}\frac{N-1-fN}{N} \simeq \underbrace{\frac{\beta}{\gamma}}_{R_0}(1-f)$$

So

$$R_{eff} = R_0(1-f)$$

To prevent the disease from becoming an epidemic, we require $R_{eff} < 1$, hence

$$R_0(1-f) < 1,$$

hence:

$$f < 1 - \frac{1}{R_0}.$$

The value $f = 1 - \frac{1}{R_0}$ is the <u>Herd Immunity Threshold</u>.