A study of $(x(q+1), x; 2, q)$*-minihypers*

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(joint work with I. Landjev)

- **Linear** $[n, k, d]$ **code** C **over** $\mathbb{F}_q$ is:
  - k-dimensional subspace of $V(n, q)$,
  - minimum distance d.

- Given k, d, q, for linear $[n, k, d]$ code over $\mathbb{F}_q$,
- **Griesmer (lower) bound**

$$n \geq \sum_{i=0}^{k-1} \lceil \frac{d}{q^i} \rceil = g_q(k, d).$$

- **Question:** do there exist $[g_q(k, d), k, d]$ codes?

THEOREM (HAMADA AND HELLESETH)

Griesmer (lower) bound
equivalent with
minihypers in finite projective spaces.

$(x(q+1), x; 2, q)$-MINIHYPERS

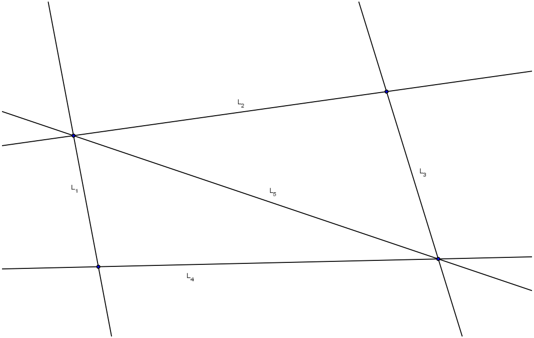
- Hill and Ward: detailed study of

$$(x(q+1), x; 2, q)\text{-minihypers.}$$

- Weighted set of $x(q+1)$ points in $\mathrm{PG}(2, q)$ intersecting every line in at least x points.
- Classical example: sum of x (not necessarily distinct) lines

$$L_1 + \cdots + L_x.$$

$(x(q+1), x; 2, q)$-MINIHYPERS



THEOREM (HILL, WARD)

Let F be an $(x(q+1), x; 2, q)$-minihyper, $q = p^m$, p prime, $m \geq 1$, with $x < q$, where p^f divides x. Then for each line L in $PG(2, q)$, $|L \cap F| \equiv x \pmod{p^{f+1}}$.

THEOREM (HILL, WARD)

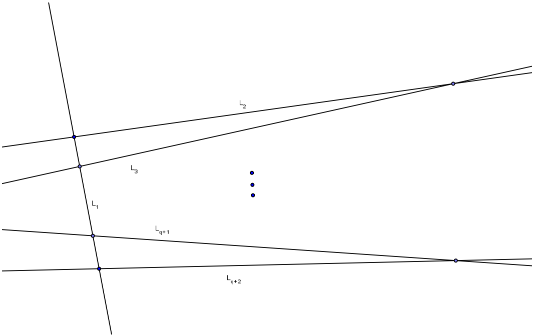
Every $(x(q+1), x; 2, q)$-minihyper F, $q = p^m$, p prime, $m \geq 1$, with $x \leq q - \frac{q}{p}$, is a sum of x lines.

THEOREM (HILL, WARD)

Every $(x(q+1), x; 2, q)$-minihyper F, q even, $m \geq 1$, with $x \leq q/2$, is a sum of x lines.

- Take a dual hyperoval $\mathcal{O}$ in $\mathrm{PG}(2, q)$, q even: $q + 2$ lines $L_1, \ldots, L_{q+2}$, no three through a common point.
- Point of $\mathrm{PG}(2, q)$ lies on zero or two lines of $\mathcal{O}$.

Dual hyperoval example



Dual hyperoval example

- Take a dual hyperoval $\mathcal{O}$ in $PG(2, q)$, q even: $q + 2$ lines, no three through a common point.
- Point of $PG(2, q)$ lies on zero or two lines of $\mathcal{O}$.
- Union of the lines of dual hyperoval is $((q/2 + 1)(q + 1), q/2 + 1; 2, q)$-minihyper.
- This particular example is not a sum of $q/2 + 1$ lines **with integer coefficients**.
- This particular example is a sum of $q + 2$ lines with coefficients $1/2$:

$$\frac{1}{2}L_1 + \cdots + \frac{1}{2}L_{q+2}.$$

Rational sum of lines

Theorem (Landjev, Storme)

Let F be an $(x(q+1), x; 2, q)$-minihyper. Then there exist lines $L_1, \ldots, L_s$ and positive rational numbers $c_1, \ldots, c_s$, such that

$$F = c_1 L_1 + \cdots + c_s L_s,$$

with $\sum_{i=1}^{s} c_i = x$.

THEOREM (LANDJEV, STORME, METSCH)

Every $((q/2+1)(q+1), q/2+1; 2, q)$*-minihyper, q even, is either:*

- *sum of* $q/2+1$ *lines,*
- *a sum*
$$\frac{1}{2}L_1 + \cdots + \frac{1}{2}L_{q+2},$$
where $\{L_1, \ldots, L_{q+2}\}$ *is dual hyperoval.*

BALL CONSTRUCTION

- Let $q = 2^m$ and consider two hyperovals H_1 and H_2 in $\mathrm{PG}(2, q)$ that meet in $q + 2 - x$ points.
- Let F be the symmetric difference of H_1 and H_2.
- $|F| = 2x$ and $|F \cap L| = 0, 2$, or 4 for every line L.
- Dualize and regard 4-lines as 2-points, 2-lines as 1-points, and 0-lines as 0-points, we get an $(x(q+1), x)$-minihyper.

- Assume multiset K with $|K| = sx$ such that $|K \cap L| = is$, $i \in \mathbb{N}$, for every line L.
- Define multiset F in the dual plane in which lines of multiplicity is become points of multiplicity i, then F is an $(x(q+1), x)$-minihyper.

EXAMPLES

- Complement in $AG(2, q)$, q even, of maximal arc,
- Complement of unital in $PG(2, q)$, q square,
- Complement of small linear blocking set in $PG(2, q)$,
- $(q + t, t)$-arc in $PG(2, q)$, q even, of type $(0, 2, t)$.

- Let K be $(q+4,4)$-arc of type $(0,2,4)$ in $\mathrm{PG}(2,q)$, q even.
- Then Ball's construction gives $((\frac{q}{2}+2)(q+1),\frac{q}{2}+2;2,q)$-minihyper.

THEOREM (LANDJEV, STORME)

Every $((\frac{q}{2}+2)(q+1), \frac{q}{2}+2; 2, q)$*-minihyper* K, q *even,* $q \geq 8$*, is either:*

(1) a sum $L_1 + \cdots + L_{q/2+2}$ *of* $q/2+2$ *lines* $L_1, \ldots, L_{q/2+2}$*, or*

(2) the sum of a line and a $((\frac{q}{2}+1)(q+1), \frac{q}{2}+1; 2, q)$*-minihyper arising from a dual hyperoval in* $PG(2, q)$*,* q *even, or*

(3) a $((\frac{q}{2}+2)(q+1), \frac{q}{2}+2; 2, q)$*-minihyper constructed via a* $(q+4, 4)$*-arc of type* $(0, 2, 4)$*.*

Definition

$(x(q+1), x; 2, q)$-minihyper F is called *indecomposable* if it cannot be represented as sum $F = F_1 + F_2$ of two other minihypers $(f_i, m_i; 2, q)$-minihypers F_i, $i = 1, 2$.

THEOREM

Let F be an indecomposable $(x(q+1), x; 2, q)$-minihyper, then

- *for every line L, $|L \cap F| \leq x + q - 1$,*
- *there is at least one point with weight zero (else $F - PG(2, q)$ is an $((x-q)(q+1) - 1, x - q - 1; 2, q)$-minihyper).*

THEOREM (LANDJEV, STORME)

Let F be an indecomposable $(x(q+1), x; 2, q)$-minihyper, then

- $x \leq q^2 - q$,
- $0 \leq w(P) \leq q - 1$.

- Let F be an indecomposable $(x(q+1), x; 2, q)$-minihyper. Let P be a point of weight zero.
- Define

$$w'(Q) = \begin{cases} q - 1 - w(Q) & \text{if } Q \neq P; \\ 0 & \text{if } Q = P. \end{cases} \quad (1)$$

- Then new $(y(q+1), y; 2, q)$-minihyper F', with $y = q^2 - q - x$, is obtained.

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Decomposability result

Theorem (Hill, Ward)

Every $(x(q+1), x; 2, q)$*-minihyper* F, $q = p^m$, p *prime,* $m \geq 1$, *with* $x \leq q - \frac{q}{p}$, *is a sum of* x *lines.*

Theorem (Landjev, Storme)

Every $(x(q+1), x; 2, q)$*-minihyper* F', $q = p^m$, p *prime,* $m \geq 1$, *with* $x \geq q^2 - 2q + \frac{q}{p}$, *is decomposable.*

- Similar results for $(x|PG(N-1,q)|, x|PG(N-2,q)|; N, q)$-minihypers.
- Similar results by K. Metsch (Universität Giessen, Germany)

Thank you for your attention!