Measures on quantum logics of idempotents matrices over finite fields

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15.07.2009

Prehistory: the Gleason theorem

We consider the real Hilbert space H with the scalar product $(\cdot, \cdot)$ and the *quantum logic* of all *projections* (i.e., *self-adjoint idempotents*) in the set of all linear operators on H. Two projections, P and Q, are said to be orthogonal iff $PQ = QP = 0$. A function μ on the set of all projections with non-negative real values is said to be a *measure* iff

$$\mu\left(\sum_n P_n\right) = \sum_n \mu(P_n) \qquad (*)$$

for any sequence or finite set (P_n) of pairwise orthogonal projections.

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Theorem 0 (A.M. Gleason, 1957). *Any measure* ($\dim(H) \geq 3$) *admits the representation*

$$\mu(P) = tr(TP) \qquad (1)$$

where T is a unique positive nuclear operator in H.

The representation (1) fails if μ is a signed measure (with values in $(-\infty, +\infty)$) and H is finite-dimensional. A construction of *counterexamples* uses existence of a function on the real line which is additive but not linear.

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The crucial case is $\dim(H) = 3$. The proof uses triples P_1, P_2, P_3 of pairwise orthogonal one-dimensional projections. Any such triple gives an equation

$$\mu(P_1) + \mu(P_2) + \mu(P_3) = \mu(Id) \tag{2}$$

connecting the values of μ.

History: signed measures on idempotents

Theorem 1 (DM, 1989). *Let us consider the quantum logic* $\mathfrak{P}(H)$ *of all continuous linear idempotents on* H *(*$\dim(H) = \infty$*). Then any finitely additive signed measure which is* σ*-additive on every* σ*-subalgebra of* $\mathfrak{P}(H)$ *admits the representation*

$$\mu(P) = tr(TP) \tag{1}$$

where T *is a unique nuclear operator in* H*.*

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Namely, we present the set $\mathfrak{P}(H)$ of all idempotents as the union

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where $\Pi_A(H)$ is the set of all idempotents which are self-adjoint w.r.t. a scalar product A.

Thus, for any scalar product A we have (1) with some $T = T_A$. So, the idea of the proof is to glue all the T_A by using

$$\Pi_A(H) \cap \Pi_B(H) \neq \emptyset.$$

Why we need extensions of the field of rationals?

The idempotent version of the Gleason theorem may be reformulated for Banach spaces (which do not have scalar products) and even for linear topological spaces.

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This brings us to the case of linear spaces over the field $\mathbf{Q}$ of rationals.

Theorem 2 (DM, 1995). *Any* $\mathbf{Q}$*-valued measure on the set of all rational idempotent* $n \times n$*-matrices* ($\dim(H) \geq 3$) *admits the representation*

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where T *is a unique rational* $n \times n$*-matrix.*

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where T *is a unique rational* $n \times n$*-matrix.*

The first variant of the *proof* used computer calculations. Now I proved this theorem without computing by using some symmetrization construction.

How to prove without computing?

Let $I_0, I_1, I_2, \ldots, I_s$ be commuting involutions on the set of idempotents and $\nu = \nu(\varepsilon_1, \ldots, \varepsilon_s)$ defined by

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$\nu(P) = \mu(P) + \sum_{i \leq s} \varepsilon_i \mu(I_i(P)) + \sum_{i<j \leq s} \varepsilon_i \varepsilon_j \mu(I_i I_j(P)) +$
$\sum_{i<j<k \leq s} \varepsilon_i \varepsilon_j \varepsilon_k \mu(I_i I_j I_k(P)) + \ldots$
where $\varepsilon_i = \pm 1$.

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$\sum_{i<j<k\le s} \varepsilon_i \varepsilon_j \varepsilon_k \mu(I_i I_j I_k(P)) + \dots$
where $\varepsilon_i = \pm 1$.

Every ν is either invariant or changes the sign of $I_i(P)$, and $\mu = 1/2^s \cdot \sum_{\varepsilon_i} \nu(\varepsilon_1, \dots, \varepsilon_s)$.

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In this direction we have no success.

The Gleason theorem and a problem for finite fields

Problem. Let $\mathbf{F}_{p^k}$ be a finite field and $\mathfrak{P}(\mathbf{F}_{p^k})$ the set of all idempotent $\mathbf{F}_{p^k}$-valued $n \times n$-matrices. Consider a function μ on $\mathfrak{P}(\mathbf{F}_{p^k})$ with values in the prime field $\mathbf{F}_p$ satisfying

$$\mu\left(\sum_{i\leq n} P_i\right) = \sum_{i\leq n} \mu(P_i)$$

for any finite set of pairwise orthogonal projections P_i. Does μ admit an additive extension to the set of *all* $\mathbf{F}_{p^k}$-valued $n \times n$-matrices $(n \geq 3)$ with trace belonging to $\mathbf{F}_p$?

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Remark. The case $n > 3$ reduces to the case $n = 3$.

Theorem 3 (DM, 1995). *The problem has the affirmative solution in the case when* $\mathbf{F}_{p^k}$ *is prime* (i.e., $k = 1$).

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Theorem 4 (DM, 1998). *The problem has the affirmative solution in the case* $p^k = 4$.

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Theorem 4 (DM, 1998). *The problem has the affirmative solution in the case* $p^k = 4$.

The first proof used computer calculations. Now I proved this theorem without computing by using some symmetrization construction. (The construction of the proof of Theorem 2 is not convenient in this case since we cannot divide by 2.)

My students (Khomutova, Skvortsov) constructed programms for the cases $p^k = 8$, $p^k = 9$. The calculation gives the affirmative answer in these cases, too. Unfortunately, I cannot verify the computing, so I am not sure.

References

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