Error-Block Codes and Poset Metrics

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- P-weight on $\mathbb{F}_q^n$:

$$\omega_P(v) = |\langle \text{supp}(v) \rangle|$$

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- Rosenbloom and Tsfasman - codes where some entries overrule others.
  The Rosenbloom-Tsfasman metric corresponds to P being a union of chains of the same size.
- New problems: given a code C, which are the posets that make C "best" in some way? I.e., perfect or MDS.

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- $V_i = \mathbb{F}_q^{k_i}$
- the π-metric on $V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$:

$$\omega_\pi(v) = |\,\mathrm{supp}_\pi(v)|$$

where $v = v_1 + \cdots + v_n, v_i \in V_i$, and
$\mathrm{supp}_\pi(v) = \{i; v_i \neq 0\}$

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- one has two "parameters" to deal with: the poset P and the dimensions $k_i = \dim V_i$.

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- (__,Panek, Firer) C is (P, π)-r-perfect $\iff$ there exists a linear function $f : V_{r+1} \oplus \ldots \oplus V_n \to V_1 \oplus \ldots \oplus V_r$ such that

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- $$H_3 \text{ is 1-perfect in } V = V_1 \oplus \cdots \oplus V_s \iff$$
  - P has only one minimal element i
  - V_i is four-dimensional
  - V_i does not contain the support of a minimal codeword

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- (__,Panek, Firer) - description of posets (not all) P such that $\mathcal{G}_{24}$ is one or two-(P, π)-perfect.

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- $\mathcal{A}$ = group of the T_σ's.

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  where $v \in V_i$ and $\text{supp}(u) \subset \langle i \rangle$;
- $\mathcal{T}$ = group of these "triangular" maps.

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- matrix form of an element of $\mathcal{T}$:

$$[T]_B = \begin{pmatrix} [T]^1_{B_1} & [T]^1_{B_2} & [T]^1_{B_3} & \cdots & [T]^1_{B_3} \\ 0 & [T]^2_{B_2} & [T]^2_{B_3} & \cdots & [T]^2_{B_3} \\ 0 & 0 & [T]^3_{B_3} & \cdots & [T]^3_{B_3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & [T]^n_{B_n} \end{pmatrix}$$

where each block $[T]^k_{B_k}$ is invertible.
This is the group of units of the associated incidence algebra (over (P, π)).

Automorphisms

- $GL_{(P,\pi)}(V) =$ automorphisms of $(V, \omega_{P,\pi})$.
  If $T \in GL_{(P,\pi)}(V)$ then $\exists! F \in \mathcal{T}$ and $\sigma \in Aut(P, \pi)$ such that

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- Moreover,

$$GL_{(P,\pi)}(V) \cong \mathcal{T} \rtimes \mathcal{A}$$

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  - $\mathcal{U} \cong S_{m_1} \times S_{m_2} \times \cdots \times S_{m_l} \subset S_n$.

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