# Equicharacteristic Galois representations of local function fields


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## Preliminaries

Let $k$ be an algebraically closed field of characteristic $p > 0$.
(Example: $\overline{\mathbb{F}_p}$.)

Let $K = k((t))$ (the field of Laurent series over $k$).

Let $\mathcal{G}_K = \mathrm{Gal}(K^{sep}/K)$.

We are concerned with $\mathcal{G}_K$-representations over the fields $\mathbb{F}_{p^r}$ ($r \geq 1$).

## Examples of representations

(1) Suppose $p \neq 2$. Let $L = K[X]/(X^2 - t)$. Then $\mathrm{Gal}(L/K) \cong \mathbb{Z}/2\mathbb{Z}$. Let $V$ be a one-dimensional $\mathbb{F}_p$-vector space on which $\mathrm{Gal}(L/K)$ acts by negation.

$V$ is a basic example of a "tame" representation of $\mathcal{G}_K$. (A representation $\rho\colon G \to \mathrm{Aut}(W)$ is tame if $p \nmid |\mathrm{im}\rho|$.)

(2) Let $n$ and $r$ be positive integers such that $n \mid (p^r - 1)$. Choose an $n$th root $t^{1/n} \in K^{sep}$. The set

$$T_{n,r} := \{ct^{1/n} \mid c \in \mathbb{F}_{p^r}\} \subseteq K^{sep}$$

forms a one-dimensional representation of $\mathcal{G}_K$.

(Note: $V \cong T_{2,1}$.)

## Examples of representations (cont.)

(3) Choose a positive integer $N$ which is coprime to $p$. Let $R = K[X]/(X^p - X - t^{-N})$.

$R$ is a field. $[R : K] = p$.

Let $\gamma\colon R \to R$ be the map which takes $X$ to $X+1$. Note that $(X+1)^p - (X+1) - t^{-N} = X^p + 1 - X - 1 - t^{-N} = X^p - X - t^{-N}$, therefore $\gamma$ is a field automorphism. $\gamma$ generates an automorphism group of size $p$.

$\mathrm{Gal}(R/K) \cong \mathbb{Z}/p\mathbb{Z}$.

Let $W_N = (\mathbb{F}_p)^2$, with an action by $\mathrm{Gal}(R/K)$ given by

$$\begin{aligned} \gamma(\mathbf{e}_1) &= \mathbf{e}_1, \\ \gamma(\mathbf{e}_2) &= \mathbf{e}_2 + \mathbf{e}_1. \end{aligned}$$

$W_N$ is a "wild" representation of $\mathcal{G}_K$.

# The minimal root index

The minimal root index ("$\mathfrak{C}$") measures the "ramification" of an $\mathbb{F}_{p^r}[\mathcal{G}_K]$-module.

**The definition of the minimal root index:**
Suppose that $W$ is an $\mathbb{F}_{p^r}[\mathcal{G}_K]$-module, (with $r \geq 1$). Let

$$\mathcal{W} = \left(W \otimes_{\mathbb{F}_{p^r}} K^{sep}\right)^{\mathcal{G}_K}$$

Let $F^r$ denote the Frobenius endomorphism,

$$\begin{aligned} F^r : W \otimes K^{sep} &\rightarrow W \otimes K^{sep}, \\ w \otimes s &\mapsto w \otimes s^{(p^r)}. \end{aligned}$$

Let $\mathcal{W}_0 \subseteq \mathcal{W}$ denote the largest $k[[t]]$-submodule such that the following two conditions are satisfied:

- $F^r(\mathcal{W}_0) \subseteq \mathcal{W}_0$, and
- The intersection of the submodules $\mathcal{W}_i := k[[t]]\left(F^{ri}(\mathcal{W}_0)\right) \subseteq \mathcal{W}$ is zero. ($i = 1, 2, 3, \ldots.$)

Then,

$$\mathfrak{C}(W) := \frac{\dim_k \mathcal{W}_0 / \mathcal{W}_1}{p^r - 1}.$$

Some properties:

(1) $\mathfrak{C}$ is additive over direct sums.

(2) $\mathfrak{C}$ is invariant under base change. ($- \mapsto - \otimes \mathbb{F}_{p^{rs}}$.)

(3) For any $m, n, r$, with $m < n$ and $n \mid (p^r - 1)$,

$$\mathfrak{C}(T_{n,r}^{\otimes m}) = 1 - \frac{m}{n}.$$

(Note: Properties (1)-(3) are sufficient to determine $\mathfrak{C}$ for any tame representation.)

(4) $\mathfrak{C}$ is superadditive over exact sequences.

## The minimal root index (cont.)

(5) The minimal root index of the wild representation $W_N$ (from Example (3)) is

$$\mathfrak{C}(W_N) = 1 + \lceil N/p^r \rceil.$$

# Motivation

**Theorem:**
Let $X \to \operatorname{Spec} k$ be a smooth projective curve of genus $g$. Let $U \subseteq X$ be an open subcurve. Let $\mathcal{H}$ be a constructible étale sheaf of $\mathbb{F}_p$-modules on $X$ which is locally constant on $U$ and zero outside of $U$. Then,

$$\sum_{i=0}^{1}(-1)^i h^i(X,\mathcal{H}) \geq (1-g)\operatorname{rank}(\mathcal{H}) - \sum_{x \in |X \smallsetminus U|} \mathfrak{C}\left(\mathcal{H}_{(x)}\right),$$

where $\mathcal{H}_{(x)}$ denotes the pullback of $\mathcal{H}$ via any morphism $\operatorname{Spec} k((t)) \to X$ determined by a local parameter at $x$.

**Ref:** *Equicharacteristic étale cohomology in dimension one*, C. Miller. http://arxiv.org/abs/0906.4093

## A Conjecture

Let $L/K$ be a finite Galois extension. Let $U$ be the regular representation of $\mathrm{Gal}(L/K)$. Let $D_{L/K}$ denote the different of the field extension $L/K$. Then I conjecture that

$$\mathfrak{C}(U) = \frac{D_{L/K}}{2} + 1.$$

Is this true?