Zeta functions in Number Theory and Combinatorics

Winnie Li

Pennsylvania State University, U.S.A.
and
National Center for Theoretical Sciences, Taiwan

1. Riemann zeta function

The Riemann zeta function is

$$\zeta(s) = \sum_{n \geq 1} n^{-s} = \prod_{p \text{ prime}} (1 + p^{-s} + p^{-2s} + \cdots)$$
$$= \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} \quad \text{for } \Re s > 1.$$

- $\zeta(s)$ has a meromorphic continuation to the whole s-plane;
- It satisfies a functional equation relating $\zeta(s)$ to $\zeta(1-s)$;
- It vanishes at $s = -2, -4, \ldots$, called trivial zeros of ζ.

Riemann Hypothesis: all nontrivial zeros of $\zeta(s)$ lie on the line of symmetry $\Re(s) = \frac{1}{2}$.

2. Zeta functions of varieties

V: smooth irred. proj. variety of dim. d defined over $\mathbb{F}_q$
The zeta function of V is

$$Z(V,u) = \exp\Big(\sum_{n\geq 1} \frac{N_n}{n} u^n\Big) = \prod_{v \text{ closed pts}} \frac{1}{(1-u^{\deg v})}$$

$$= \frac{P_1(u)P_3(u)\cdots P_{2d-1}(u)}{P_0(u)P_2(u)\cdots P_{2d}(u)}.$$

Here $N_n = \#V(\mathbb{F}_{q^n})$ and each $P_i(u)$ is a polynomial in $\mathbb{Z}[u]$ with constant term 1.

RH: the roots of $P_i(u) (\neq 1)$ have absolute value $q^{-i/2}$.
For a curve, RH says all zeros of $Z(V;q^{-s})$ lie on $\Re s = \frac{1}{2}$.
Proved by Hasse and Weil for curves, and Deligne for varieties in general.

3. The Ihara zeta function of a graph

- X : connected undirected finite graph
- Want to count tailless geodesic cycles.

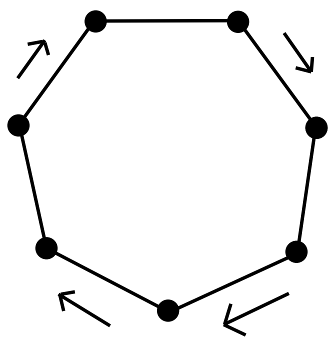

Figure 1: without tail

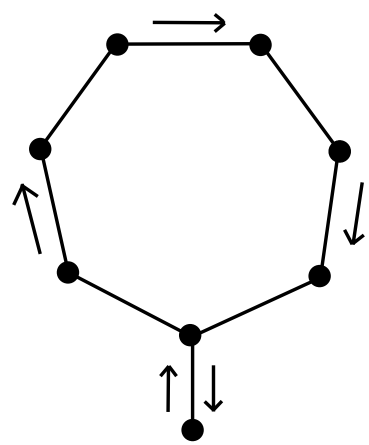

Figure 2: with tail

- A cycle has a starting point and an orientation.
- A cycle is *primitive* if it is not obtained by repeating a cycle (of shorter length) more than once.

- Two cycles are *equivalent* if one is obtained from the other by shifting the starting point.

The Ihara zeta function of X is defined as

$$Z(X;u) = \prod_{[C]} \frac{1}{1-u^{l(C)}} = \exp\left(\sum_{n\geq 1} \frac{N_n}{n} u^n\right),$$

where $[C]$ runs through all equiv. classes of primitive geodesic and tailless cycles C, $l(C)$ is the length of C, and N_n is the number of geodesic tailless cycles of length n.

4. Properties of zeta functions of regular graphs

Ihara (1968): Let X be a finite $(q+1)$-regular graph with n vertices. Then its zeta function $Z(X,u)$ is a rational function of the form

$$Z(X;u) = \frac{(1-u^2)^{\chi(X)}}{\det(I - Au + qu^2 I)} ,$$

where $\chi(X) = n - n(q+1)/2 = -n(q-1)/2$ is the Euler characteristic of X and A is the adjacency matrix of X.

If X is not regular, replace qI by Q, the degree matrix minus the identity matrix on vertices–Bass, Stark-Terras, Hoffman.

- The trivial eigenvalues of X are $\pm(q+1)$, of multiplicity one.
- X is called a *Ramanujan graph* if the nontrivial eigenvalues λ satisfy the bound

$$|\lambda| \leq 2\sqrt{q}.$$

- X is Ramanujan if and only if $Z(X,u)$ satisfies RH, i.e. the nontrivial poles of $Z(X,u)$ all have absolute value $q^{-1/2}$.
- Let $\{X_j\}$ be a family of $(q+1)$-regular graphs with $|X_j| \to \infty$. Alon-Boppana :

$$\liminf_{j\to\infty} \max_{\lambda\neq q+1 \text{ of } X_j} \lambda \geq 2\sqrt{q}.$$

Li, Serre : if X_j contains few short cycles of odd length,

$$\limsup_{j\to\infty} \min_{\lambda\neq -q-1 \text{ of } X_j} \lambda \leq -2\sqrt{q}.$$

5. The Hashimoto edge zeta function of a graph

Endow two orientations on each edge of a finite graph X. The neighbors of $u \to v$ are the directed edges $v \to w$ with $w \neq u$. Associate the edge adjacency matrix A_e.

Hashimoto (1989): $N_n = \mathrm{Tr} A_e^n$ so that

$$Z(X, u) = \frac{1}{\det(I - A_e u)}.$$

Combined with Ihara's Theorem, we have

$$\frac{(1-u^2)^{\chi(X)}}{\det(I - Au + qu^2 I)} = \frac{1}{\det(I - A_e u)}.$$

6. Connections with number theory

When $q = p^r$ is a prime power, let F be a local field with the ring of integers $\mathcal{O}_F$ and residue field $\mathcal{O}_F/\pi\mathcal{O}_F$ of size q.

$$\mathcal{T} \quad = \quad \mathrm{PGL}_2(F)/\mathrm{PGL}_2(\mathcal{O}_F)$$

$$\text{vertices} \quad \leftrightarrow \quad \mathrm{PGL}_2(\mathcal{O}_F)\text{-cosets}$$

$$\text{vertex adjacency operator } A \quad \leftrightarrow \quad \text{Hecke operator on}$$

$$\mathrm{PGL}_2(\mathcal{O}_F) \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \mathrm{PGL}_2(\mathcal{O}_F)$$

$$\text{directed edges} \quad \leftrightarrow \quad \mathcal{I}\text{-cosets } (\mathcal{I} = \text{Iwahori subgroup})$$

$$\text{edge adjacency operator } A_e \quad \leftrightarrow \quad \text{Iwahori-Hecke operator on}$$

$$\mathcal{I} \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \mathcal{I}$$

$X = X_\Gamma = \Gamma\backslash\mathrm{PGL}_2(F)/\mathrm{PGL}_2(\mathcal{O}_F) = \Gamma\backslash\mathcal{T}$ for a torsion free discrete cocompact subgroup Γ of $PGL_2(F)$.

- Ihara (1968): A torsion free discrete cocompact subgroup Γ of $\mathrm{PGL}_2(F)$ is free of rank $1-\chi(X_\Gamma)$.
- Take $F = \mathbb{Q}_p$ so that $q = p$, $\mathcal{O}_F = \mathbb{Z}_p$, and let

  D_ℓ= the definite quaternion algebra over $\mathbb{Q}$ ramified only at ∞ and prime $\ell \neq p$.

  $\Gamma_\ell = D_\ell^\times(\mathbb{Z}[\frac{1}{p}])$ mod center is a discrete subgroup of $\mathrm{PGL}_2(\mathbb{Q}_p)$ with compact quotient; torsion free if $\ell \equiv 1 \pmod{12}$.
- $X_{\Gamma_\ell} = \Gamma_\ell\backslash\mathrm{PGL}_2(\mathbb{Q}_p)/\mathrm{PGL}_2(\mathbb{Z}_p)$ is a Ramanujan graph.
- $\frac{\det(I-Au+pu^2I)}{(1-u)(1-pu)}$ from X_{Γ_ℓ} is the numerator of the zeta function of the modular curve $X_0(\ell)$ mod p.

- The class number of the quaternion algebra D_ℓ $= |X_{\Gamma_\ell}| = 1 + g_0(\ell)$, where $g_0(\ell)$ is the genus of $X_0(\ell)$.

- The number of spanning trees in a graph X is called the class number $\kappa(X)$ of the graph.

  Hashimoto: For a $(q+1)$-regular graph X,

$$\kappa(X)|X| = \frac{\det(I - Au + qu^2 I)}{(1-u)(1-qu)}\Big|_{u=1}.$$

- When $X = X_{\Gamma_\ell}$, this gives the relation

$$\kappa(X_{\Gamma_\ell})(1 + g_0(\ell)) = |Jac_{X_0(\ell)}(\mathbb{F}_p)|.$$

7. The Bruhat-Tits building attached to $\mathbf{PGL}_3(F)$

- $G = PGL_3(F)$, $K = PGL_3(\mathcal{O}_F)$. The Bruhat-Tits building $\mathcal{B}_3 = G/K$ is a 2-dim'l simplicial complex.
- Have a filtration

$$K \supset E \text{ (parahoric subgroup) } \supset B \text{ (Iwahoric subgroup)}$$

- vertices $\leftrightarrow$ K-cosets

  Each vertex gK has a type $\tau(gK) = \operatorname{ord}_\pi \det g \pmod 3$.
- The type of an edge $gK \to g'K$ is $\tau(g'K) - \tau(gK) = 1$ or 2. Call $g'K$ a type 1 or 2 neighbor of gK, accordingly.
- type one edges $\leftrightarrow$ E-cosets
- directed chambers $\leftrightarrow$ B-cosets.

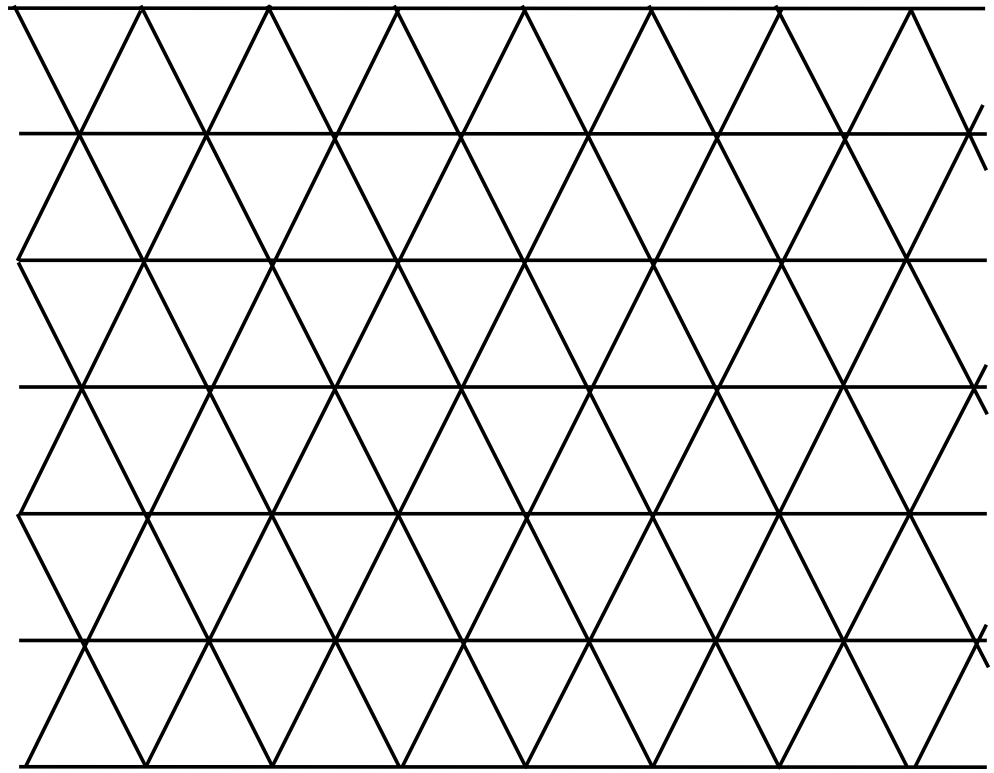

Figure 3: the fundamental apartment

8. Operators on $\mathcal{B}_3$

- A_i = the adjacency matrix of type i neighbors of vertices $(i = 1, 2)$

  A_1 and A_2 are Hecke operators on certain K-double cosets.

- L_E = the type one edge adjacency matrix

  It is a parahoric operator on an E-double coset.

  Its transpose L_E^t is the type two edge adjacency matrix.

- L_B = the adjacency matrix of directed chambers.

  It is an Iwahori-Hecke operator on a B-double coset.

9. Ramanujan complexes

The spectrum of the adjacency operator on the $(q+1)$-regular tree is $[-2\sqrt{q},\ 2\sqrt{q}]$. A $(q+1)$-regular Ramanujan graph has its nontrivial eigenvalues fall in the spectrum of its universal cover.

The building $\mathcal{B}_3$ is $(q+1)$-regular and simply connected; it is the universal cover of its finite quotients. The operators A_1 and A_2 on $\mathcal{B}_3$ have the same spectrum

$$\Omega := \{q(z_1 + z_2 + z_3)\ :\ z_j \in S^1,\ z_1 z_2 z_3 = 1\}.$$

A finite quotient X of $\mathcal{B}_3$ is called Ramanujan if the eigenvalues of A_1 and A_2 on X fall in Ω. Spectrally optimal.

Similar definition for Ramanujan complexes arising from the building $\mathcal{B}_n$ attached to $PGL_n(F)$.

Margulis (1988), Lubotzky-Phillips-Sarnak (1988), Mestre-Oesterlé (1986), Pizer (1990), Morgenstern (1994) : explicit constructions of infinite families of $(q+1)$-regular Ramanujan graphs for $q = p^a$

Li (2004): obtained infinite families of $(q + 1)$-regular Ramanujan complexes by taking quotients of $\mathcal{B}_n$ by discrete cocompact torsion-free subgroups of G.

Lubotzky-Samuels-Vishne (2005): Not all such subgroups yield Ramanujan complexes.

Explicit constructions of Ramanujan complexes given by Lubotzky-Samuels-Vishne (2005) and Sarveniazi (2007).

10. Finite quotients of $\mathcal{B}_3$

The finite complexes we consider are $X_\Gamma = \Gamma\backslash G/K = \Gamma\backslash\mathcal{B}_3$, where Γ is a torsion free cocompact discrete subgroup of G and $\mathrm{ord}_\pi \det \Gamma \subset 3\mathbb{Z}$ so that Γ identifies vertices of the same type.

Division algebras of degree 3 yield many such Γ's.

Goal: Define a suitable zeta function, which counts geodesic cycles up to homotopy in X_Γ, and which has the following properties:

- It is a rational function with a closed form expression;
- It provides topological and spectral information of the complex;
- The complex is Ramanujan if and only if its zeta satisfies RH.

Joint work with Ming-Hsuan Kang

11. Zeta function of the complex X_Γ

The zeta function of X_Γ is defined to be

$$Z(X_\Gamma, u) = \prod_{[\mathfrak{C}]} \frac{1}{1 - u^{l(\mathfrak{C})}} = \exp\Big(\sum_{n \geq 1} \frac{N_n u^n}{n}\Big),$$

where $[\mathfrak{C}]$ runs through the equiv. classes of primitive tailless geodesic cycles in X_Γ consisting of only type one edges or type two edges, $l(\mathfrak{C})$ is the length of the cycle C, and N_n is the number of tailless geodesic cycles of length n.

Observe that

$$Z(X_\Gamma, u) = \frac{1}{\det(I - L_E u)} \frac{1}{\det(I - L_E^t u^2)}.$$

12. Main results for 2-dim'l complex zeta functions

Main Theorem (Kang-L.)

(1) *$Z(X_\Gamma, u)$ is a rational function given by*

$$Z(X_\Gamma, u) = \frac{(1-u^3)^{\chi(X_\Gamma)}}{\det(I - A_1 u + qA_2 u^2 - q^3 u^3 I)\det(I + L_B u)},$$

where $\chi(X_\Gamma) = \#V - \#E + \#C$ is the Euler characteristic of X_Γ.

(2) *X_Γ is a Ramanujan complex if and only if $Z(X_\Gamma, u)$ satisfies RH.*

Remarks. (1) Ramanujan complexes defined in terms of the eigenvalues of A_1 and A_2 can be rephrased as

(i) the nontrivial zeros of $\det(I - A_1 u + qA_2 u^2 - q^3 u^3 I)$ have absolute value q^{-1}.

Kang-L-Wang showed that this has two more equivalent statements:

(ii) the nontrivial zeros of $\det(I - L_E u)$ have absolute values q^{-1} and $q^{-1/2}$;

(iii) the nontrivial zeros of $\det(I - L_B u)$ have absolute values 1, $q^{-1/2}$ and $q^{-1/4}$.

(2) The zeta identity can be reformulated in terms of operators:

$$\frac{(1-u^3)^{\chi(X_\Gamma)}}{\det(I - A_1 u + qA_2 u^2 - q^3 u^3 I)} = \frac{\det(I + L_B u)}{\det(I - L_E u)\det(I - L_E^t u^2)},$$

compared with the identity for graphs:

$$\frac{(1-u^2)^{\chi(X)}}{\det(I - Au + qu^2 I)} = \frac{1}{\det(I - A_e u)}.$$

(3) By regarding A_1 and A_2 as operators acting on $L^2(\Gamma\backslash G/K)$, L_E on $L^2(\Gamma\backslash G/E)$ and L_B on $L^2(\Gamma\backslash G/B)$, Kang-L-Wang gave a representation theoretic proof of (2).

(4) Similar connections to zeta functions of modular surfaces.

13. Comparison between graphs and 2-dimensional complexes

Let Γ be a discrete torsion-free cocompact subgroup of $PGL_n(F)$.
$X_\Gamma = \Gamma \backslash PGL_n(F)/PGL_n(\mathcal{O}_F)$

Case $n = 2$.

- every element in Γ has two eigenvalues in F;
- the primitive equivalence classes $[C]$ of geodesic tailless cycles $\leftrightarrow$ the primitive conjugacy classes $[\gamma]$ of Γ;
- $l(C) = \mathrm{ord}_\pi b$, where $1, b$ are eigenvalues of γ with $\mathrm{ord}_\pi b \geq 0$.

Case $n = 3$.

- Every element in Γ has one or three eigenvalues in F;
- Γ contains elements having only one eigenvalue in F;
- A primitive conjugacy class $[\gamma]$ contains finitely many primitive $[C]$, and a non-primitive conjugacy class $[\gamma]$ may also contain finitely many primitive $[C]$;
- $l(C) = m + n$, where $\gamma \in K diag(1, \pi^n, \pi^{m+n})K$, $n, m \geq 0$.

14. Zeta functions of finite quotients of $PGL_n(F)$

- $G = PGL_n(F)$, $K = PGL_n(\mathcal{O}_F)$. The Bruhat-Tits building $\mathcal{B}_n = G/K$ is an $(n-1)$-dimensional simplicial complex, simply connected and $(q+1)$-regular.
- G acts transitively on simplices of the same dimension.
- vertices of $\mathcal{B}_n \leftrightarrow K$-cosets.
- A_i, $i = 1, ..., n-1$ (Hecke) operators on vertices.
- A k-simplex together with a choice of one of its vertices is called a directed k-simplex.
- For $0 \leq k \leq n-1$, let $v(k) = diag(1, ..., 1, \pi, ..., \pi)K$, where π occurs k times.

- A standard directed k-dim'l simplex has vertices $v(0), v(i_1), ...,$ $v(i_k)$, where $0 < i_1 < \cdots < i_k < n$, and chosen vertex $v(0)$. The gap vector $\mathbf{a} = (i_1 - 0, i_2 - i_1, ..., i_k - i_{k-1})$ is called its type.

  The stabilizer of these vertices is the subgroup $P_{\mathbf{a}}$ of K.

- Directed k-simplices of type $\mathbf{a} \leftrightarrow P_{\mathbf{a}}$-cosets.

- For each type $\mathbf{a}$, there is an adjacency matrix $A_{\mathbf{a}}$ on directed k-simplices of type $\mathbf{a}$; $A_{\mathbf{a}}$ is also expressed as an operator on certain $P_{\mathbf{a}}$-double coset.

- For $\mathbf{a} = (a_1, ..., a_k)$, set $l(\mathbf{a}) = k$, $|\mathbf{a}| = a_1 + \cdots + a_k$, and its primitive part $pr(\mathbf{a}) = (a_1, ..., a_s)$ to be the shortest initial segment such that $\mathbf{a}$ is $pr(\mathbf{a})$ repeated k/s times.

Theorem (Ming-Hsuan Kang)

Let Γ *be a discrete torsion-free cocompact subgroup of* $PGL_n(F)$ *and* $X_\Gamma = \Gamma \backslash \mathcal{B}_n$. *The following zeta identity on* X_Γ *holds:*

$$\frac{(1-u^n)^{\chi(X_\Gamma)}}{\det(\sum_{i=0}^{n}(-1)^i q^{i(i-1)/2} A_i u^i)}$$
$$= \prod_{[\mathbf{a}]} \det(I - A_{\mathbf{a}}((-1)^{l(\mathbf{a})+l(pr(\mathbf{a}))}u)^{|pr(\mathbf{a})|})^{(-1)^{l(\mathbf{a})}}.$$

Here $\chi(X_\Gamma)$ *is the Euler characteristic of* X_Γ, $A_0 = A_n = I$, *and* $\mathbf{a}$ *runs through all types up to cyclic permutations.*

Ex. For $n = 4$ the identity is

$$\frac{(1-u^4)^{\chi(X_\Gamma)}}{\det(I - A_1 u + qA_2 u^2 - q^3 A_3 u^3 + q^6 I u^4)}$$
$$= \frac{\det(I + A_{(1,1)}u)\det(I - A_{(1,2)}u^3)}{\det(I - A_{(1)}u)\det(I - A_{(2)}u^2)\det(I - A_{(3)}u^3)\det(I - A_{(1,1,1)}u)}.$$