# Algebraic continued fractions in power series fields

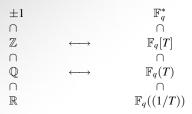
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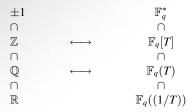
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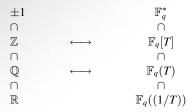
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For instance:  $T/(T-1) = 1 + T^{-1} + \dots + T^{-n} + \dots$  in  $\mathbb{F}_p(T)$  $\alpha \in \mathbb{F}_q(T) \iff (a_n)_{n \le k}$  is eventually periodic.

• Every  $\alpha \in \mathbb{F}(q)$  can be expanded as a continued fraction :

 $\alpha = a_0 + 1/(a_1 + 1/(a_2 + 1/(\dots = [a_0, a_1, a_2, \dots]))$ 

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The expansion is finite  $\iff \alpha \in \mathbb{F}_q(T)$ . For instance :  $(T^2 - 1)^2/(2T^3 + 4T) = [3T, 2T, 3T, 2T]$  in  $\mathbb{F}_5(T)$ .

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For  $\alpha \in \mathbb{F}(q) \setminus \mathbb{F}_q(T)$  we define (the approximation exponent):

$$\nu(\alpha) = 2 + \limsup_{n>0} (\deg(a_{n+1}) / \sum_{0 \le i \le n} \deg(a_i)).$$

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 $\alpha = (A\alpha^r + B)/(C\alpha^r + D)$ 

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- $\alpha \in \mathbb{F}(q)$  is hyperquadratic if  $\alpha \in \mathbb{H}(q) = \bigcup_{t \ge 0} \mathbb{H}_r(q)$ .
- $\alpha \in \mathbb{H}(q) \Rightarrow \alpha' = b_0 + b_1 \alpha + b_2 \alpha^2$  where  $b_i \in \mathbb{F}_q(T)$ .  $\alpha$  is hyperquadratic  $\Rightarrow \alpha$  is "differential-quadratic".

# Two basic examples in $\mathbb{F}(p)$

• Example 1.  $\alpha = [T, T^r, T^{r^2}, \dots, T^{r^k}, \dots].$ Hence  $\alpha = T + 1/\alpha^r$  and  $\alpha \in \mathbb{H}_r(p)$  for all p.

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• **Example 2.** K. Mahler (1949)  $\alpha = 1/T + 1/T^r + \dots + 1/T^{r^k} + \dots$ Hence  $\alpha = 1/T + \alpha^r$  and  $\alpha \in \mathbb{H}_r(p)$  for all p. We put  $\beta = 1/\alpha = [a_1, a_2, \dots, a_n, \dots]$ . For r > 2, we have  $\beta = [T, \beta_2]$  and  $\beta^r = -T^2\beta_2 - T$ . Two basic examples in  $\mathbb{F}(p)$ 

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The sequence  $(a_i)_{i\geq 1}$  is defined recursively by  $a_1 = T$  and :

 $a_{4k+1} = T$   $a_{4k+2} = -a_{2k+1}^r/T^2$   $a_{4k+3} = -T$   $a_{4k+4} = a_{2k+2}^r$  for  $k \ge 0$ .

We observe that here we also have  $\nu(\alpha) = r$ .

►  $\alpha \in \mathbb{F}(q)$  and  $[\mathbb{F}_q(T, \alpha) : \mathbb{F}_q(T)] = n \Rightarrow \nu(\alpha) \in [2; n].$ Liouville (1849) - Mahler (1949).

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- ►  $\alpha \in \mathbb{F}(q) \setminus \mathbb{H}(q)$  and  $[\mathbb{F}_q(T, \alpha) : \mathbb{F}_q(T)] = n \Rightarrow \nu(\alpha) \leq [n/2] + 1.$ Osgood (1975) - de Mathan-L. (1996).

## Generating continued fractions in $\mathbb{H}(q)$

#### Proposition (L. 2006)

Let  $l \ge 1$  be an integer. Let  $(a_1, a_2, ..., a_l) \in (\mathbb{F}_q[T])^l$  and  $(R, P, Q) \in (\mathbb{F}_q[T])^3$  with  $\deg(Q) < \deg(P) < r$ . Then there exists a unique  $\alpha \in \mathbb{H}_r(q)$  such that

 $\alpha = [a_1, a_2, \ldots, a_l, \alpha_{l+1}]$  and  $R\alpha^r = P\alpha_{l+1} + Q$ .

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If  $(a_1, a_2, ..., a_l, R, P, Q) \in (\mathbb{F}_q[T])^{l+3}$  are well chosen then the continued fraction expansion can be given explicitly.

In  $\mathbb{F}(p)$  for all p, we have

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$$\alpha^{13} = (T^2 + 8)^4 \alpha_7 + 4 \int_0^T (x^2 + 8)^3 dx.$$

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From this we can deduce : There exists a sequence  $(\lambda_n)_{n\geq 1}$  in  $\mathbb{F}_{13}^*$  such that

$$a_n = \lambda_n A_{\nu_9(8n-2)}$$
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where the sequence  $(A_m)_{m\geq 0}$  in  $\mathbb{F}_q[T]$  is given by

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Here we have  $\nu(\alpha) = 8/3$ .

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where the sequence  $(\delta_n)_{n\geq 1}$  in  $\mathbb{F}_{27}^*$  satisfies  $\delta_1 = u^4$  and

$$\delta_{3n-1} = u^{5(-1)^n} \delta_n^3 (1+\gamma_n^3), \quad \delta_{3n} = \frac{\gamma_n^3 - 1}{\delta_{3n-1}}, \quad \delta_{3n+1} = \frac{\delta_{3n-1}}{1-\gamma_n^6} \quad \text{for} \quad n \ge 1$$

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and finally the sequence  $(\gamma_n)_{n\geq 1}$  in  $\mathbb{F}_{27}^*$  satisfies  $\gamma_1 = u$  and

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#### A comment on Mathematics

"Roughly speaking, unfashionable mathematics consists of those parts of mathematics which were declared by the mandarins of Bourbaki not to be mathematics. A number of very beautiful mathematical discoveries fall into this category. To be mathematics according to Bourbaki, an idea should be general, abstract, coherent, and connected by clear logical relationships with the rest of mathematics. Excluded from mathematics are particular facts, concrete objects which just happen to exist for no identifiable reason, things which a mathematician would call accidental or sporadic.....things of accidental beauty, special functions, particular number fields, ..... "

Freeman J. Dyson