The maximum number of rational points on plane curves over a finite field (arXiv:0907.1325)

Seon Jeong Kim
Gyeongsang National University, Korea

(Joint work with Masaaki Homma)

Fq9

UCD, July 13-17, 2009

Notations

$\mathbb{F}_q$, a finite field with q elements

$\mathbb{P}^2$, the projective plane over $\bar{\mathbb{F}}_q$, the algebraic closure of $\mathbb{F}_q$

C, the curve defined by a homogeneous equation $f(x,y,z)=0$ with coefficients in $\mathbb{F}_q$

$\mathbb{P}^2(\mathbb{F}_q) := \{(\alpha,\beta,\gamma) \in \mathbb{P}^2 \mid \alpha,\beta,\gamma \in \mathbb{F}_q\}$

$C(\mathbb{F}_q) := \{(\alpha,\beta,\gamma) \in \mathbb{P}^2(\mathbb{F}_q) \mid f(\alpha,\beta,\gamma)=0\}$,
the set of $\mathbb{F}_q$-rational points of C

$N_q(C)$, the cardinality of the set $C(\mathbb{F}_q)$.

Preliminaries

We suppose that C has no $\mathbb{F}_q$-line as a component.

$M_q(d) := \max\{N_q(C) \mid C \in \mathscr{C}_d(\mathbb{F}_q)\}$,

where $\mathscr{C}_d(\mathbb{F}_q)$ is the set of all plane curves over $\mathbb{F}_q$ of degree d without an $\mathbb{F}_q$-linear component.

$M_q(d) \leq {}^{\#}\mathbb{P}^2(\mathbb{F}_q) = q^2 + q + 1$ for any $d \geq 1$

For $d \geq q + 2$, $M_q(d) = q^2 + q + 1$. (Homma and Kim)

In particular, for $d = q + 2$, G. Tallini proved there are irreducible curves C with $N_q(C) = q^2 + q + 1$. Homma and I proved there are even nonsingular curves C with $N_q(C) = q^2 + q + 1$.

$M_q(d) \leq (d-1)q+1$ **(Sziklai Conjecture)**

For $d = q+2$, $M_q(q+1) = q^2+q+1$. (Tallini)

For $d = q+1$, $M_q(q+1) = q^2+1$. (Homma and Kim)

For $d = \sqrt{q}+1$, when q is a square, $M_q(d) = (d-1)q+1$ is attained for a Hermitian curve. (well-known)

For $d = 3$, $M_q(3) = 2q+1$ if and only if $q = 2$ or 3 or 4. (Schoof)

For $d = 2$, $M_q(2) = q+1$. (well-known)

Some bounds and a conjecture

$M_q(d) \leq (d-1)q + \left\lfloor \frac{d}{2} \right\rfloor$ (B. Segre)

$M_q(d) \leq (d-1)q + (q+2-d)$ (Homma and Kim)

This bound is better than Segre's in the range $\frac{2}{3}q + \frac{5}{3} < d \leq q+1$ and implies that the Sziklai conjecture is true for $d = q+1$.

For $d = q = 4$, the Sziklai's conjecture is false since $M_4(4) = 14 (> (4-1)4+1)$. Indeed, $N_4(C) = 14$ for the nonsingular curve C defined by the equation $X^4+Y^4+Z^4+X^2Y^2+Y^2Z^2+Z^2X^2+X^2YZ+XY^2Z+XYZ^2 = 0.$

Modified Sziklai's conjecture

Unless C is a curve defined over $\mathbb{F}_4$ which is projectively equivalent to

$$X^4+Y^4+Z^4+X^2Y^2+Y^2Z^2+Z^2X^2+X^2YZ+XY^2Z+XYZ^2=0 \quad (1)$$

over $\mathbb{F}_4$, we might have

$$N_q(C) \le (d-1)q+1. \quad (2)$$

The Main Theorems

Theorem 1

For $d = q$, the modified Sziklai's conjecture is true, and for each q there exists a nonsingular curve of degree q over $\mathbb{F}_q$ with $(q-1)q+1$ rational points.

Now the case $d \leq q-1$ is remained .

Theorem 2

The modified Sziklai's conjecture is true for nonsingular curves of degree $d \leq q-1$. Moreover there is an example of a nonsingular curve for which equality holds in (2) if $d = q+2, q+1, q, q-1, \sqrt{q}+1$ (when q is square), or 2.

In this talk, we concentrate on a proof of Theorem 1.

For $q = 2$, we have $M_2(2) = 3$, since $M_q(2) = q + 1$ for arbitrary q.

For $q = 3$, $M_3(3) = 7$ (Segre's bound and an example).

For $q = 4$, $M_4(4) = 14$ (Segre's bound and an example), and we already proved that any plane curve attaining this bound is projectively equivalent to the curve defined by (1) over $\mathbb{F}_4$.

Thus it remains to prove the theorem for $q \geq 5$.

Lemma 1 and 2

Lemma 1 (Homma and Kim)

If $2 \leq d \leq q+1$, then

$$M_q(d) \leq (d-1)q + (q+2-d).$$

In particular, we have $M_q(q) \leq (q-1)q + 2$.

Lemma 2

Let C be the plane curve defined by the equation $x^q - xz^{q-1} + y^{q-1}z - z^q$ over $\mathbb{F}_q$ where $q \geq 2$. Then C is nonsingular and ${}^{\#}C(\mathbb{F}_q) = (q-1)q + 1$.

To prove Theorem 1

Thus, to prove $M_q(q) \leq (q-1)q+1$ it suffices to prove that there is no irreducible curve C of degree q with $N_q(q) = (q-1)q+2$.

Indeed, note that if the curve C of degree q is reducible and decomposed into two curves of degree d_1 and d_2 with $2 \leq d_1, d_2 \leq q-2$ as $C = C_1 \cup C_2$, then

$$\begin{aligned} N_q(C) &\leq N_q(C_1) + N_q(C_2) \\ &\leq ((d_1-1)q + \lfloor \frac{d_1}{2} \rfloor) + ((d_2-1)q + \lfloor \frac{d_2}{2} \rfloor) \\ &\leq (q-1)q + \lfloor \frac{q}{2} \rfloor - q \leq (q-1)q. \end{aligned}$$

More notations

Let f be a homogeneous polynomial in $\mathbb{F}_q[x, y, z]$.

$Z(f) := \{(\alpha, \beta, \gamma) \in \mathbb{P}^2(\mathbb{F}_q) \mid f(\alpha, \beta, \gamma) = 0\}$, the zero set of f

We use the following notation:

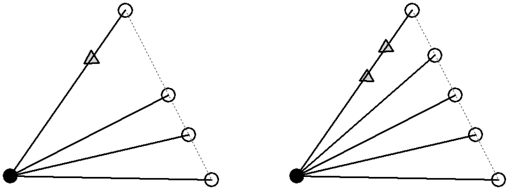
(α, β, γ) denotes a point.
$[\alpha, \beta, \gamma]$ denotes the line with equation $\alpha x + \beta y + \gamma z = 0$.

Note that the lines through the origin $(0, 0, 1)$ have the equation of the form $\alpha x + \beta y = 0$, i.e., are expressed as $[\alpha, \beta, 0]$.

Lemma 3

Lemma 3

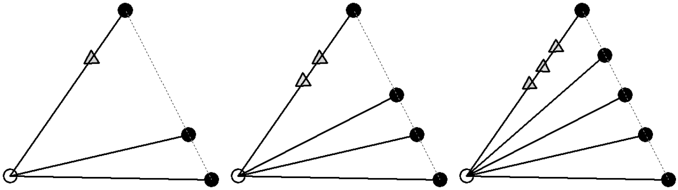
Let f be an irreducible homogeneous polynomial of degree $d = q \geq 5$ in $\mathbb{F}_q[x, y, z]$. Let $[-\gamma_i, \beta_i, 0]$, $i = 1, \ldots, k+1$ with $3 \leq k \leq q$ be $k+1$ distinct lines through the origin $(0, 0, 1)$. Suppose that $Z(f) \supseteq [-\gamma_i, \beta_i, 0] - \{(\beta_i, \gamma_i, 0)\}$ for $i = 1, \ldots, k$.
If $Z(f)$ contains $q+2-k$ points in the deleted line $[-\gamma_{k+1}, \beta_{k+1}, 0] - \{(\beta_{k+1}, \gamma_{k+1}, 0)\}$, then $Z(f)$ contains all of them.



Lemma 4

Lemma 4

Let f be an irreducible homogeneous polynomial of degree $d = q \geq 5$ in $\mathbb{F}_q[x, y, z]$. Let $[-\gamma_i, \beta_i, 0]$, $i = 1, \ldots, k+1$ with $2 \leq k \leq q-1$ be $k+1$ distinct lines through the origin $(0, 0, 1)$. Suppose that $Z(f) \supseteq [-\gamma_i, \beta_i, 0] - \{(0, 0, 1)\}$ for $i = 1, \ldots, k$.
If $Z(f)$ contains $q+1-k$ points in the deleted line $[-\gamma_{k+1}, \beta_{k+1}, 0] - \{(0, 0, 1)\}$, then $Z(f)$ contains all of them.



Proof of Theorem 1

Suppose that there exists an irreducible curve C of degree q with $N_q(C) = (q-1)q + 2$. If C is singular at some $\mathbb{F}_q$-rational point P, then each line through P meets C at most $q-1$ points. Then $N_q(C) \leq (q-2)\cdot(q+1)+1 = q^2 - q - 1$. Thus we may assume that C is nonsingular at every $\mathbb{F}_q$-rational point of C.

Let a_i $(0 \leq i \leq q)$ be the number of lines (i-point lines) ℓ in the projective plane such that ${}^{\#}\ell \cap C(\mathbb{F}_q) = i$. Then we obtain the following;

(1) $\sum_{i=0}^{q} a_i = q^2 + q + 1$ (the number of all lines on the plane).

(2) $\sum_{i=0}^{q} i a_i = (q^2 - q + 2) \cdot (q+1)$ (the sum of $^{\#}\ell \cap C(\mathbb{F}_q)$ for all lines on the plane).

(3) If q is even [resp. odd], then

$$\sum_{i=1}^{\frac{q}{2}-1} i a_i + \sum_{i=\frac{q}{2}}^{q} (q-i) a_i \geq q^2 - q + 2$$

$$\text{[resp. } \sum_{i=1}^{\frac{q-1}{2}} i a_i + \sum_{i=\frac{q+1}{2}}^{q} (q-i) a_i \geq q^2 - q + 2\text{]}.$$

(the number of tangent lines at $\mathbb{F}_q$-rational points to C).

(4) $\sum_{i=2}^{q} \binom{i}{2} a_i = \binom{q^2-q+2}{2}$ (counting the number of elements in the set $\{(\{P, Q\}, \langle P, Q\rangle) \mid P, Q \in C(\mathbb{F}_q) \text{ and } P \neq Q\}$ in two ways).

From above equations (1), (2), (3) and (4), we obtain $qa_0 + (q-2)a_1 + (q-4)a_2 + \cdots \leq q-4$, which implies $a_0 = 0$ and $a_1 = 0$ since a_i's are nonnegative integers.
Now we need the following lemma.

Lemma 5

At least one among $\{a_i \mid 2 \leq i \leq q-3\}$ is non-zero.

Proof of Lemma 5. Suppose that all of them are zero. Then the equations become

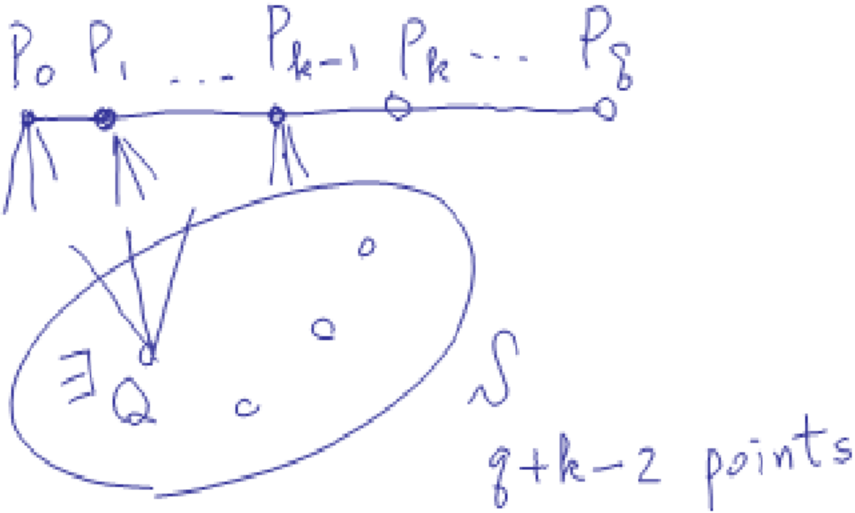
(1) $a_{q-2} + a_{q-1} + a_q = q^2 + q + 1$

(2) $(q-2)a_{q-2} + (q-1)a_{q-1} + qa_q = (q^2 - q + 2) \cdot (q+1)$

(4) $\binom{q-2}{2} a_{q-2} + \binom{q-1}{2} a_{q-1} + \binom{q}{2} a_q + = \binom{q^2-q+2}{2}$

Using the elimination method, we have $a_{q-1} = -(q-3)^2 + 1$, which is smaller than zero for $q \geq 5$, and hence it is contradiction.

Now let k be the smallest positive integer such that $a_k > 0$. By Lemma 5, we have $2 \leq k \leq q-3$. Let ℓ_0 be a fixed k-point line. Let $\ell_0 \cap \mathbb{P}^2(\mathbb{F}_q) = \{P_0, P_1, \ldots, P_q\}$, and $\ell_0 \cap C(\mathbb{F}_q) = \{P_0, P_1, \ldots, P_{k-1}\}$. Let $S := \mathbb{P}^2(\mathbb{F}_q) - \ell_0 - C(\mathbb{F}_q)$ then ${}^{\#}S = q + k - 2$. For each P_i with $0 \leq i \leq k-1$, let $\mathscr{S}(P_i)$ be the set of points $Q \in S$ such that the line $\langle P_i, Q \rangle$ is a q-point line. Then ${}^{\#}\mathscr{S}(P_i) \geq q - k + 2$ since the union of q lines except ℓ_0 through P_i contains S.


P0 P1 ... Pk-1 Pk ... Pg
∃Q
S
g+k-2 points

Now we consider the case $k \geq 3$ at first. Since we have

$$\sum_{i=0}^{k-1} {}^{\#}\mathscr{S}(P_i) \geq k(q-k+2) > 2(q+k-2) = 2 \cdot {}^{\#}S$$

for $3 \leq k \leq q-3$ by simple computation, there exists a point $Q \in S$ such that $Q \in \mathscr{S}(P_{i_1}) \cap \mathscr{S}(P_{i_2}) \cap \mathscr{S}(P_{i_3})$ for some distinct $i_1, i_2, i_3 \in \{0, 1, \ldots, k-1\}$. Then we have a contradiction by the following lemma which can be proved using Lemma 4.

Lemma 6

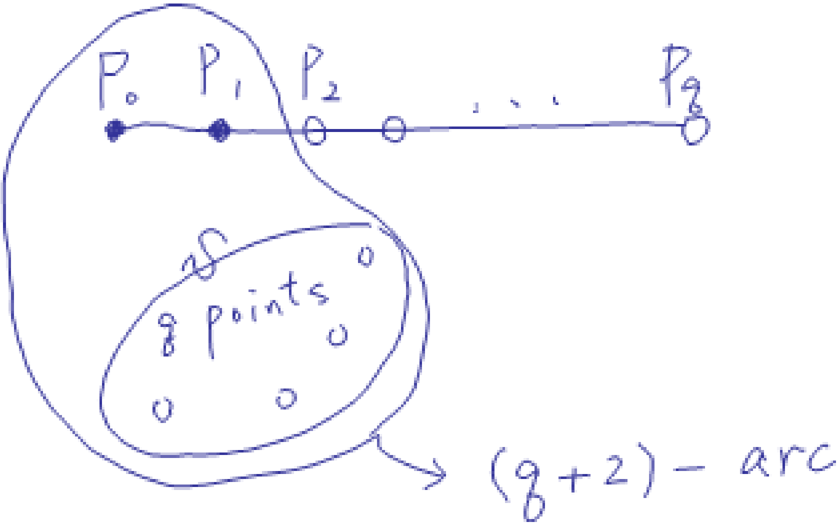
Let $Q \in \mathbb{P}^2(\mathbb{F}_q) - C(\mathbb{F}_q)$. If there are at least two q-point lines containing Q, then the pencil of lines through Q consists of two q-point lines and $q-1$ $(q-2)$-point lines.

Proof. Suppose that there are exactly $r(\geq 2)$ q-point lines through Q. Then by Lemma 4, each of the other lines through Q contains at most $q-r$ rational points of C. Note that $r \leq q-1$ since $a_0 = a_1 = 0$. The total number of rational points on C is equal to $\sum_{Q\in\ell} {}^{\#}(\ell \cap C(\mathbb{F}_q))$. Thus

$$q^2 - q + 2 = \sum_{Q\in\ell} {}^{\#}(\ell \cap C(\mathbb{F}_q)) \leq rq + (q-r+1)(q-r),$$

which is equivalent to $(r-2)q \leq (r-2)(r+1)$. Since $2 \leq r \leq q-1$, that inequality implies $r = 2$ or $r = q-1$. If $r = q-1$, at least one of the remaining two lines is 0-point line or 1-point line which contradicts the fact $a_0 = a_1 = 0$. Thus we have $r = 2$ and the other $q-1$ lines are exactly $(q-2)$-point ones.

The case $a_2 = 1$



Now only the case $a_2 > 0$ is remained. In fact, the computation above Lemma 5 implies $a_2 = 1$. As in the part of proof above Lemma 6, we use the same notation. The line ℓ_0 is the unique 2-point line and $\ell_0 \cap C(\mathbb{F}_7) = \{P_0, P_1\}$,
$\ell_0 \cap (\mathbb{P}^2(\mathbb{F}_q) - C(\mathbb{F}_q)) = \{P_i \mid 2 \leq i \leq q\}$. Then every line through P_0 or P_1 except ℓ_0 is a q-point line.
Let $\ell_1, \ldots, \ell_q$ be the q lines through P_0 except ℓ_0.
Let $\ell_i \cap (\mathbb{P}^2(\mathbb{F}_q) - C(\mathbb{F}_q)) = \{Q_i\}$ for $i = 1, \ldots, q$, then $S = \{Q_1, Q_2, \ldots, Q_q\}$.
By Lemma 3, no three points of S are collinear. Thus the set $S \cup \{P_0, P_1\} = \{P_0, P_1, Q_1, Q_2, \ldots, Q_q\}$ becomes a $(q+2)$-arc, i.e., no three points in that set are collinear. For odd q, this is contradiction since $\mathbb{P}^2(\mathbb{F}_q)$ can not contain r-arcs for $r \geq q + 2$. Thus we may assume q is even.

The case $a_2 = 1$ and q even

Since $S \cup \{P_0, P_1\}$ is $(q+2)$-arc, i.e., a hyperoval, every line in the plane is an 2-secant line or 0-secant line of it. Counting the number of points in $S \cup \{P_0, P_1\}$ implies that exactly $\frac{q}{2}$ lines through P_q (or any P_i, $2 \le i \le q$) are 0-secant lines, equivalently q-point lines. Since $q \ge 5$, in fact $q \ge 8$, we have a contradiction using Lemma 6 again.

Thus there is no irreducible plane curve C of degree q over $\mathbb{F}_q$ with $N_q(C) = q^2 - q + 2$. By combining the fact mentioned below Lemma 2, we conclude there is no plane curve C of degree q with no $\mathbb{F}_q$-linear component with $N_q(C) = q^2 - q + 2$.

Therefore $M_q(q) = q^2 - q + 1$ for $q \ge 5$. Thus the proof of Theorem 1 is complete.

A proof of Theorem 2

Now we may assume that C is a nonsingular plane curve over $\mathbb{F}_q$ of degree d with $1 < d \leq q-1$. We prove that $N_q(C) \leq (d-1)q+1$. A nonsingular plane curve C defined over $\mathbb{F}_q$ is said to be q-Frobenius nonclassical if $F_q(P) \in T_P(C)$ for a general $\overline{\mathbb{F}}_q$-point P, where F_q is the q-th power Frobenius map and $T_P(C)$ is the embedded tangent line at P to C. Stöhr and Voloch showed that if C is q-Frobenius classical of degree d, then

$$N_q(C) \leq \frac{1}{2}d(d+q-1), \tag{3}$$

and Hefez and Voloch proved that if C is q-Frobenius nonclassical of degree d, then $d \geq \sqrt{q}+1$ and

$$N_q(C) = d(q-d+2). \tag{4}$$

Each of these two estimates for $N_q(C)$ is stronger than the expected bound if $2 \leq d \leq q-1$ for (3) or $d \geq \sqrt{q}+1$ for (4). In fact,

$$(d-1)q+1-\frac{1}{2}d(d+q-1)=\frac{1}{2}(d-2)(q-d-1)$$

and

$$(d-1)q+1-d(q-d+2)=(d-\sqrt{q}-1)(d+\sqrt{q}-1).$$

□

References

[1] A. Hefez and J. F. Voloch, Frobenius nonclassical curves, Arch. Math. (Basel) 54 (1990) 263–273; Correction, Arch. Math. (Basel) 57 (1991) 416.

[2] J. W. P. Hirschfeld, Projective Geometries over Finite Fields, Clarendon Press, Oxford (1998).

[3] M. Homma and S. J. Kim, *Nonsingular plane filling curves of minimum degree over a finite field and their automorphism groups: Supplements to a work of Tallini*, Preprint 2008.

[4] M. Homma and S. J. Kim, *Around Sziklai's conjecture on the number of points of a plane curve over a finite field*, preprint 2008.

[5] R. Schoof, Nonsingular plane cubic curves over finite fields, J. Combin. Theory Ser. A 46 (1987) 183–211.

[6] B. Segre, *Le geometrie di Galois*, Ann. Mat. Pura Appl. (4) 48 1–96 (1959).

[7] M. P. Sziklai, *A bound on the number of points of a plane curve*, Finite Fields Appl. 14 (2008) 41–43.

[8] K.-O. Stöhr and J. F. Voloch, *Weierstrass points and curves over finite fields*, Proc. London Math. Soc. (3) 52 (1986), 1–19.