Arcs in Galois ring planes invariant under a Singer cycle

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Outline

Galois Rings

The uniform case $m = 2$

Let $q = p^r > 1$ be a prime power.

Galois rings of length 2

$\mathbb{G}_q := \mathbb{Z}_{p^2}[X]/(h)$, where $h \in \mathbb{Z}_{p^2}[X]$ is monic of degree r and irreducible mod p.

The polynomial h (whose particular choice does not matter) may be chosen as a divisor of $X^{q-1} - 1$.

Basic properties of $\mathbb{G}_q$

- $\mathbb{G}_q \supset (p) \supset \{0\}$ are the ideals of $\mathbb{G}_q$;
- $\mathbb{G}_q/(p) \cong \mathbb{F}_q$ (as rings) and $(p) \cong \mathbb{F}_q$ as $\mathbb{F}_q$-spaces;
- $|\mathbb{G}_q| = q^2$;
- $\mathrm{Char}(\mathbb{G}_q) = p^2$.

The last property characterizes $\mathbb{G}_q$ among the chain rings of length 2 with residue field $\mathbb{F}_q$. (The other r such rings have characteristic p.)

Galois Ring Planes

The projective Hjelmslev plane over $\mathbb{G}_q$

$\mathrm{PHG}(2,\mathbb{G}_q) := (\mathcal{P},\mathcal{L},\subseteq)$, where $\mathcal{P}$ ("points") and $\mathcal{L}$ ("lines") denote the sets of free rank 1 (resp., free rank 2) submodules of $\mathbb{G}_q^3$ (or any other free $\mathbb{G}_q$-module of rank 3).

Since $\mathbb{G}_q$ is commutative, we need not distinguish between left and right projective Hjelmslev planes over $\mathbb{G}_q$.

Basic properties of $\mathrm{PHG}(2,\mathbb{G}_q)$

- $\mathrm{PHG}(2,\mathbb{G}_q)$ is a symmetric divisible design with parameters $(m,n,k,\lambda_1,\lambda_2) = (q^2+q+1,q^2,q^2+q,q,1)$.
- Lines intersect point classes in either 0 or q points, and dually points are on either 0 or q lines of each line class.

Thus $\mathrm{PHG}(2,\mathbb{G}_q)$ is an example of a *uniform* projective Hjelmslev plane of order q.

Galois Ring Planes (Cont'd)

$$\overline{\mathcal{P}} = \{[p]; p \in \mathcal{P}\}, \quad \overline{\mathcal{L}} = \{[L]; L \in \mathcal{L}\}$$

Structural properties of $\mathrm{PHG}(2, \mathbb{G}_q)$

- $(\overline{\mathcal{P}}, \overline{\mathcal{L}}, \overline{I}) \cong \mathrm{PG}(2, \mathbb{F}_q)$
- $[p] \cong \mathrm{AG}(2, \mathbb{F}_q)$
- $[L] \cong \mathrm{AG}(2, \mathbb{F}_q)^* \cong \mathrm{PG}(2, \mathbb{F}_q) \setminus \{p\}$

Nonempty point sets of the form $L \cap [p]$ are called *line segments*.
All line segments have size q.
The lines of $[p]$, as well as the points of $[L]$ are line segments.

Singer's Classical Theorem

Theorem (Singer 1938)

$\mathrm{PG}(2,\mathbb{F}_q)$ *admits a cyclic collineation group G acting regularly (i.e. sharply transitive) on the points (and lines) of* $\mathrm{PG}(2,\mathbb{F}_q)$.

This leads to a simplified representation $\mathrm{PG}(2,\mathbb{F}_q) \cong (\mathcal{P}', \mathcal{L}', \in)$ with

$$\mathcal{P}' := G \cong \mathbb{Z}_{q^2+q+1}, \quad \mathcal{L}' := \{D+i; i \in \mathbb{Z}_{q^2+q+1}\},$$

where $D := \{g \in S; g(p_0) \in L_0\}$ for some fixed point-line pair $(p_0, L_0) \in \mathcal{P} \times \mathcal{L}$.

Proof.

Represent the ambient space $\mathbb{F}_q^3$ as $\mathbb{F}_{q^3}$ (the cubic extension field of $\mathbb{F}_q$). Choose a primitive element β of $\mathbb{F}_{q^3}$ and consider the collineation σ induced by $\mathbb{F}_{q^3} \to \mathbb{F}_{q^3}$, $x \mapsto \beta x$ (which has the same order q^2+q+1 as the quotient group $\mathbb{F}_{q^3}^\times / \mathbb{F}_q^\times$). Let $G = \langle \sigma \rangle$. $\square$

Singer's Theorem for $\mathrm{PHG}(2, \mathbb{G}_q)$

Theorem (Hale-Jungnickel 1978)

$\mathrm{PHG}(2, \mathbb{G}_q)$ *admits a regular collineation group* $G \cong \mathbb{Z}_{q^2+q+1} \times (\mathbb{F}_q, +) \times (\mathbb{F}_q, +)$.

Proof.

Represent the ambient space $\mathbb{G}_q^3$ as $\mathbb{G}_{q^3}$ (the cubic Galois extension of $\mathbb{G}_q$). Here we use the fact that $\mathbb{G}_{q^3}$ is a free $\mathbb{G}_q$-module of rank 3.

- $\mathbb{G}_q x \in \mathcal{P}$ iff $x \in \mathbb{G}_{q^3} \setminus p\mathbb{G}_{q^3} = \mathbb{G}_{q^3}^\times$
- $\mathbb{G}_q x = \mathbb{G}_q y$ iff $y \in \mathbb{G}_q^\times x$

This implies that

$$G := \mathbb{G}_{q^3}^\times / \mathbb{G}_q^\times \cong \mathbb{Z}_{q^2+q+1} \times (\mathbb{F}_q, +) \times (\mathbb{F}_q, +)$$

acts regularly on $\mathcal{P}$. $\square$

Coordinate-Free Representation

In the classical case the trace $\mathrm{Tr} = \mathrm{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}$ serves this purpose via the trace form

$$\mathbb{F}_{q^3} \times \mathbb{F}_{q^3} \to \mathbb{F}_q, \quad (x,y) \mapsto \mathrm{Tr}(xy)$$

and the associated polarity

$$\mathcal{P} \to \mathcal{L}, \quad \mathbb{F}_q x \mapsto (\mathbb{F}_q x)^{\perp} = \{y \in \mathbb{F}_{q^3}; \mathrm{Tr}(xy) = 0\}.$$

Using Singer's Theorem this simplifies to

$$\begin{aligned}
p_i &= \mathbb{F}_q \beta^i = \sigma^i(p_0),\\
L_i &= \{x \in \mathbb{F}_{q^3}; \mathrm{Tr}(\beta^{-i}x) = 0\} = \sigma^i(L_0),\\
p_i \in L_j \iff i - j \in D &= \{i \in \mathbb{Z}_{q^2+q+1}; p_i \in L_0\}\\
&= \{i \in \mathbb{Z}_{q^2+q+1}; \mathrm{Tr}(\beta^i) = 0\}.
\end{aligned}$$

Coordinate-Free Representation (Cont'd)

Teichmüller coordinates

$$\mathrm{T}_q^* = \{x \in \mathbb{G}_q^*; x^{q-1} = 1\} \qquad \text{(Teichmüller group)}$$
$$\mathrm{T}_q = \{x \in \mathbb{G}_q; x^q = x\} = \mathrm{T}_q^\times \cup \{0\} \qquad \text{(Teichmüller set)}$$

Every $x \in \mathbb{G}_q$ has a unique representation $x = x_0 + px_1$ with $x_0, x_1 \in \mathrm{T}_q$.

Trace of the extension $\mathbb{G}_{q^3}/\mathbb{G}_q$

The trace $\mathrm{Tr} = \mathrm{Tr}_{\mathbb{G}_{q^3}/\mathbb{G}_q} : \mathbb{G}_{q^3} \to \mathbb{G}_q$ is defined by

$$\mathrm{Tr}(x_0 + px_1) = \sum_{i=0}^{2} \left(x_0^{q^i} + px_1^{q^i} \right).$$

Tr is $\mathbb{G}_q$-linear, onto, and $\ker(\mathrm{Tr})$ contains no nonzero ideal of $\mathbb{G}_{q^3}$.

Coordinate-Free Representation (Cont'd)

Theorem

(i) For every line L of $\mathrm{PHG}(\mathbb{G}_{q^3})$ *there exists a unit* $\alpha \in \mathbb{G}_{q^3}^{\times}$ *such that*

$$L = \left\{\mathbb{G}_q x \in \mathcal{P}; \mathrm{Tr}(\alpha x) = 0\right\}.$$

(ii) The equations $\mathrm{Tr}(\alpha x) = 0$ *and* $\mathrm{Tr}(\beta x) = 0$ *(*$\alpha, \beta \in \mathbb{G}_{q^3}^{\times}$*) determine the same line iff* $\alpha = u\beta$ *for some unit* $u \in \mathbb{G}_q^{\times}$*.*

Sketch of proof.

(i) Consider the $\mathbb{G}_q$-module $M = \mathrm{Hom}_{\mathbb{G}_q}(\mathbb{G}_{q^3}, \mathbb{G}_q)$ (the dual of the $\mathbb{G}_q$-module $\mathbb{G}_{q^3}$).

M is also a module over $\mathbb{G}_{q^3}$ relative to the action $(\phi a)(x) := \phi(ax)$ ($\phi \in \mathrm{Hom}_{\mathbb{G}_q}(\mathbb{G}_{q^3}, \mathbb{G}_q)$, $a, x \in \mathbb{G}_{q^3}$).

Show that M is freely generated by Tr as a $\mathbb{G}_{q^3}$-module.

Proof cont'd.

(ii) Since Tr is $\mathbb{G}_q$-linear, $\mathrm{Tr}(\alpha x) = 0$ and $\mathrm{Tr}(\beta x) = 0$ determine the same line whenever $\alpha\mathbb{G}_q^\times = \beta\mathbb{G}_q^\times$.

"Only if" then follows from $|\mathcal{P}| = q^2(q^2 + q + 1) = |\mathbb{G}_{q^3}^\times / \mathbb{G}_q^\times|$. □

Remark

The trace form $(x, y) \mapsto \mathrm{Tr}(xy)$ has an associated orthogonal polarity $\tau\colon \mathcal{P} \cup \mathcal{L} \to \mathcal{P} \cup \mathcal{L}$, sending a free rank 1 or rank 2 submodule U of the $\mathbb{G}_q$-module $\mathbb{G}_{q^3}$ to $U^\perp := \{x \in \mathbb{G}_{q^3}; \mathrm{Tr}(xy) = 0 \text{ for all } y \in U\}$. For any point $\mathbb{G}_q\alpha \in \mathcal{P}$ we have $\tau(\mathbb{G}_q\alpha) = L_\alpha$, where $L_\alpha \in \mathcal{L}$ denotes the line with equation $\mathrm{Tr}(\alpha x) = 0$.

Collineations of $\mathrm{PHG}(2, \mathbb{G}_q)$

Fundamental Theorem of Projective Hjelmslev Geometry

$\operatorname{Aut}\mathrm{PHG}(2, \mathbb{G}_q) \cong \mathrm{P\Gamma L}(3, \mathbb{G}_q)$, a group of order $r \cdot q^{11}(q^3-1)(q^2-1)$.

Projectivities of $\mathrm{PHG}(2, \mathbb{G}_q)$

$\mathrm{PGL}(3, \mathbb{G}_q) = \mathrm{GL}(3, \mathbb{G}_q)/\mathbb{G}_q^{\times}$, a group of order $q^{11}(q^3-1)(q^2-1)$.

Notes

- Collineations preserve the partitions $\overline{\mathcal{P}}$, $\overline{\mathcal{L}}$ of $\mathcal{P}$ resp. $\mathcal{L}$ into neighbour classes, as well as the sets of line segments resp. dual line segments.
- $\mathrm{PGL}(3, \mathbb{G}_q)$ acts regularly on ordered quadrangles of $\mathrm{PHG}(2, \mathbb{G}_q)$, i. e. sets of 4 points which form a quadrangle in the quotient plane $\mathrm{PG}(2, \mathbb{F}_q)$

Multisets in $\mathrm{PHG}(2,\mathbb{G}_q)$ Arising from a Singer Cycle

Singer Cycles of $\mathrm{PHG}(2,\mathbb{G}_q)$

A collineation σ of $\mathrm{PHG}(2,\mathbb{G}_q)$ is called a *Singer cycle*, if σ has order q^2+q+1 and permutes the point classes $[p] \in \overline{\mathcal{P}}$ in one cycle.

Example

If $\mathrm{T}_{q^3}^{\times} = \langle\eta\rangle$, then $\sigma\colon \mathcal{P} \to \mathcal{P}$, $\mathbb{G}_q x \mapsto \mathbb{G}_q \eta x$ (and dually for lines) defines a Singer cycle of $\mathrm{PHG}(2,\mathbb{G}_q) = \mathrm{PHG}(\mathbb{G}_{q^3})$.

Proposition

The Singer cycles generate a single conjugacy class of subgroups of $\mathrm{PGL}(3,\mathbb{G}_q)$.

Hence we need only consider a fixed Singer cycle σ.

Multisets Arising from a Singer cycle (Cont'd)

We are interested in multisets (of points) in $\mathrm{PHG}(2,\mathbb{G}_q)$ invariant under a Singer cycle σ.

Observation

A σ-invariant multiset $\mathfrak{K}$ in $\mathrm{PHG}(2,\mathbb{G}_q)$ is complete determined by its restriction $\mathfrak{k}$ to the point class $[\mathbb{G}_q 1] \cong \mathrm{AG}(2,\mathbb{F}_q)$.

Definition

We say that $\mathfrak{K}$ *is induced by* $\mathfrak{k}$.

Proposition

Suppose that the multisets $\mathfrak{K}_1$, $\mathfrak{K}_2$ in $\mathrm{PHG}(2,\mathbb{G}_q)$ are induced by multisets $\mathfrak{k}_1$, resp. $\mathfrak{k}_2$ in $[\mathbb{G}_q 1]$.

(i) *If $\mathfrak{K}_1$ and $\mathfrak{K}_2$ are equivalent then $\mathfrak{k}_1$, $\mathfrak{k}_2$ are equivalent.*

(ii) *If $\mathfrak{k}_1$, $\mathfrak{k}_2$ are translation-equivalent (i. e. there exists a translation τ of $[\mathbb{G}_q 1]$ such that $\mathfrak{k}_2 = \mathfrak{k}_1 \circ \tau$), then $\mathfrak{K}_1$ and $\mathfrak{K}_2$ are equivalent.*

σ-Invariant Arcs

First recall the following

Definition

A multiset $\mathfrak{K}$ in $\mathrm{PHG}(2,\mathbb{G}_q)$ is said to be a (k,n)-*arc* if

(i) $|\mathfrak{K}| := \mathfrak{K}(\mathcal{P}) := \sum_{p\in\mathcal{P}} \mathfrak{K}(p) = k$, and

(ii) $\mathfrak{K}(L) := \sum_{p\in L} \mathfrak{K}(p) \leq n$ for every line $L \in \mathcal{L}$.

Notes and questions

- Our goal is to find *maximal arcs* (i.e. those which maximize k for a given n) or, more generally,*complete arcs* (i.e. those which cannot be extended without increasing n).
- Restricting attention to σ-invariant arcs simplifies the construction problem. (However, we may loose something.)
- Can we compute the line multiplicities $\mathfrak{K}(L)$ from data about the inducing multiset $\mathfrak{k}$? (The answer is a somewhat restricted "yes".)
- Which multisets in $\mathrm{AG}(2,\mathbb{F}_q)$ give rise to good arcs?

The Teichmüller Set

The simplest example of a σ-invariant multiset

Definition

The multiset in $\mathrm{PHG}(2, \mathbb{G}_q)$ induced by $\{\mathbb{G}_q 1\}$ is called *Teichmüller set* and denoted by $\mathfrak{T}$.

We have $\mathfrak{T} = \{\mathbb{G}_q \eta^i ; 0 \leq i \leq q^2 + q\}$.

Up to equivalence $\mathfrak{T}$ is just the multiset induced by a single point in $[\mathbb{G}_q 1] \cong \mathrm{AG}(2, \mathbb{F}_q)$.

Theorem (H.-Landjev, 2005)

$\mathfrak{T}$ is a maximal $(q^2 + q + 1, 2)$-arc ("hyperoval") iff q is even.

Sketch of proof.

σ is transitive on line classes

$\Longrightarrow$ It suffices to check line multiplicities in one particular line class, say $[\mathrm{Tr}(x) = 0]$. This class consists of the lines $\mathrm{Tr}\big((1 + pv)x\big) = 0$, where $v \in (\mathbb{F}_{q^3}, +)/(\mathbb{F}_q, +)$.

A direct computation using the isomorphism $\mathbb{G}_q \cong \mathrm{W}_2(\mathbb{F}_q)$ (*Witt vectors of length* 2 *over* $\mathbb{F}_q$) yields the result. $\square$

Viewing Everything Within $\mathrm{AG}(2,\mathbb{F}_q)$

Consider again fixed line class, say $[L]=[\mathrm{Tr}(x)=0]$.
Define $I\subset\{0,1,\dots,q^2+q\}$ by

$$I:=\{i;\mathrm{Tr}(\eta^i)\equiv 0 \pmod p\}=\{i;[\mathbb{G}_q\eta^i] \text{ is on } [L]\}.$$

σ^i induces an isomorphism $[\mathbb{G}_q 1]\cong[\mathbb{G}_q\eta^i]$ of affine planes $\implies\sigma^i$ maps a unique parallel class of lines in $[\mathbb{G}_q 1]$ onto the parallel class of lines in $[\mathbb{G}_q\eta^i]$ having direction $[L]$.

Observations

- Putting these maps together gives a bijection ι from the set of lines of $[\mathbb{G}_q 1]\cong\mathrm{AG}(2,\mathbb{F}_q)$ to the set of line segments incident with $[L]$ (points of $[L]$).
- Under ι every line $L'\in[L]$ corresponds to a set of $q+1$ lines of $\mathrm{AG}(2,\mathbb{F}_q)$ having different directions. The q^2 sets $\iota^{-1}(L')$, $L'\in[L]$, form a single translation equivalence class.
- Knowing one such set $\iota^{-1}(L)$ enables us to compute the spectrum of $\mathfrak{k}$ w.r.t. to the whole class, and in turn the spectrum of $\mathfrak{K}$.

Observations cont'd

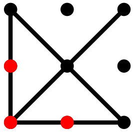
- The spectrum of $\mathfrak{T}$ w.r.t. the lines in $[L]$ equals the spectrum of $\iota^{-1}(L)$ w.r.t. to the points in $[\mathbb{G}_q 1]$.

Example

We give a computer-free proof of the existence of a (maximal) $(39,5)$-arc $\mathfrak{K}$ in the plane over $\mathbb{G}_3 = \mathbb{Z}_9$.

$|\mathfrak{K}| = 3 \cdot 13 = 3 \times \#$points of $\mathrm{PG}(2,\mathbb{F}_3)$
suggests inducing $\mathfrak{K}$ from a triangle in $\mathrm{AG}(2,\mathbb{F}_3)$.

An easy hand computation (omitted) reveals that $\mathfrak{T}$ is a (proper) 3-arc. Hence $\iota^{-1}(L)$ consists of 3 lines L_1, L_2, L_3 in $\mathrm{AG}(2,\mathbb{F}_3)$ passing through a common point p and another line L_4 with $p \notin L_4$ and $L_4 \nparallel L_i$ for $1 \leq i \leq 3$.



There exists a triangle $\mathfrak{k}$ in $\mathrm{AG}(2,\mathbb{F}_3)$ meeting $\iota^{-1}(L)$ in at most 5 points. The multiset in $\mathrm{PHG}(2,\mathbb{Z}_9)$ induced by $\mathfrak{k}$ is the required $(39,5)$-arc.

Generalization

The $(39, 5)$-arc meets every line in either 2 or 5 points.

Theorem

Let q be an odd prime. There exist $\left(\frac{1}{2}(q^4 - q), \frac{1}{2}(q^2 + q - 2)\right)$-arcs in $\mathrm{PHG}(2, \mathbb{G}_q)$ with only two line multiplicities $\frac{1}{2}(q^2 \pm q - 2)$.

Proof.

Use Singer induction from so-called triangle sets in $\mathrm{AG}(2, \mathbb{F}_q)$.

Triangle Sets

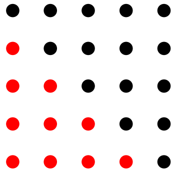
Definition

Suppose q is prime. A set of points of $\mathrm{AG}(2,\mathbb{F}_q)$ is called a *triangle set* if it is affinely equivalent to

$$\Delta := \left\{(x,y)\in\mathbb{F}_q^2; x+y<q-1\right\}.$$

Example

A triangle set in $\mathrm{AG}(2,\mathbb{F}_5)$.



Theorem

(i) *The characteristic function of a triangle set in* $\mathrm{AG}(2,\mathbb{F}_q)$ *is constant on* $q-2$ *parallel classes of lines and takes the values* $0,1,\ldots,q-1$ *exactly once on each of the remaining 3 parallel classes of lines.*

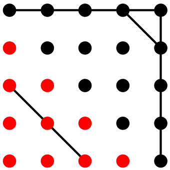
(ii) *If S is a set of points of* $\mathrm{AG}(2,\mathbb{F}_q)$ *with the foregoing properties, then q is necessarily prime and S is a triangle set.*

Observations

- The mod-q-value t of the sum of the multiplicities of any 3 lines in the 3 exceptional directions depends only on the translation equivalence class.
- The sum is 2-valued precisely for $t = q - 2$ (Example on the right) and $t = q - 1$ (3 concurrent lines).

Example

A triangle set in $\mathrm{AG}(2, \mathbb{F}_5)$.



Proof cont'd.

Induct from a triangle set $\mathfrak{k}$ such that the 3 exceptional lines of $\iota^{-1}(L)$ have $t = q - 2$ (i.e. multiplicities $q - 2$ and $2q - 2$). $\square$

Remark

Sets of points whose characteristic function is constant on all but three parallel classes of lines exist also in the affine planes $\mathrm{AG}(2, \mathbb{F}_q)$, $q = 2^r$, and give rise to arcs with two line multiplicities in the corresponding Hjelmslev planes $\mathrm{PHG}(2, \mathbb{G}_q)$.