Classification of plane curves with infinitely many Galois points

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Plan

I. What is a Galois point ?

II. How many Galois points are there ?

III. Classification of plane curves with
infinitely many Galois points

I. What is a *Galois point* ?

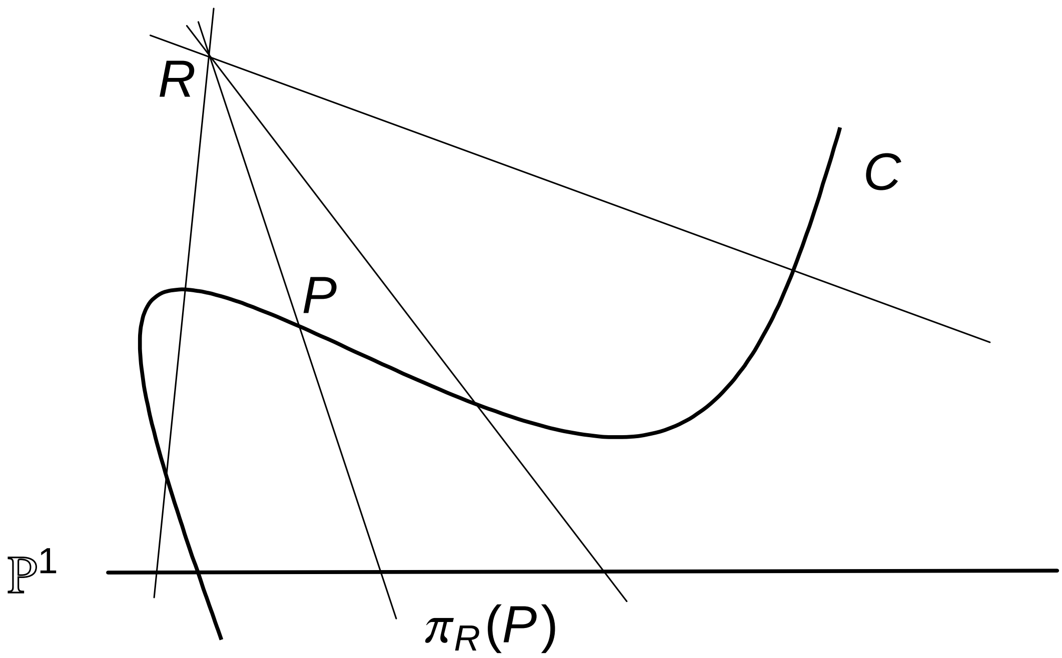
K: alg. closed field

p = char $K \geq 0$

$C \subset \mathbb{P}^2$: irred. plane curve of deg. $d \geq 3$

$R \in \mathbb{P}^2$: point

$\pi_R : C \dashrightarrow \mathbb{P}^1$; projection from R



Definition (Hisao Yoshihara, 1996)

R: Galois point (for C)

$\Leftrightarrow K(C)/\pi_R^* K(\mathbb{P}^1)$: Galois extension

Example

$p \neq 2, 3$

$C \subset \mathbb{P}^2 : X^3Z + Y^4 + Z^4 = 0$

$R_1 = (1:0:0) \in C$

$R_2 = (0:1:0) \in \mathbb{P}^2 \setminus C$

Galois points

The reason that R_1 is Galois

$\pi_{R_1} = (Y:Z) = (y:1) : C \dashrightarrow \mathbb{P}^1$

$K(C)/K(\mathbb{P}^1) = K(x, y)/K(y)$: $x^3 + y^4 + 1 = 0.$

cyclic extension

Rem. $\pi_R : C \dashrightarrow \mathbb{P}^1$; point projection

(1) $P \in C_{\mathrm{sm}} \setminus \{R\}$

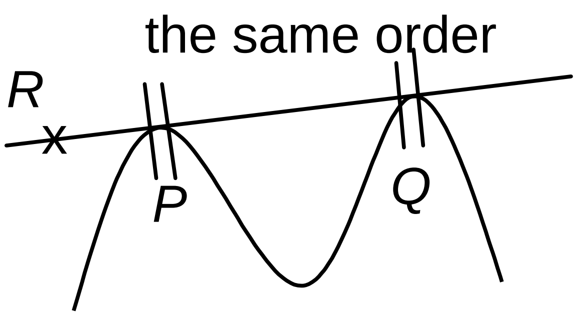
$\Rightarrow e_P = I_P(C, \overline{RP})$

[Rami. Index = Intersect. Multi.]

(2) R: Galois

$P, Q \in C_{\mathrm{sm}} \setminus \{R\}$ s.t. $\pi_R(P) = \pi_R(Q)$

$\Rightarrow e_P = e_Q$



If R is Galois

$\Rightarrow R$: the intersect. pt of multi. tangent lines

(3) $p > 0$, π_R is NOT separable

$\Rightarrow R \in T_P C$ for $\forall P \in C_{\mathrm{sm}}$

R is called a strange center.

C is said to be strange if $\exists$ strange center.

II. How many Galois points are there ?

Notation

$\Delta(C) = \{R \in C_{sm} \mid R : \text{Galois}\}$

$\Delta'(C) = \{R \in \mathbb{P}^2 \setminus C \mid R : \text{Galois}\}$

Case A: $p = 0$ & C: smooth: Completely determined

Theorem (Yoshihara)

$p = 0$, $C \subset \mathbb{P}^2$: smooth, deg. $d \geq 4$

(1) $\sharp\Delta(C) = 0, 1$ or 4.

(2) $\sharp\Delta'(C) = 0, 1$ or 3.

Case B: $p = 0$ & C: singular: Unsoloved

However, we can prove:

Proposition

The number of Galois points is finite if $p = 0$.

Case C: $p > 0$

How about a Hermitian curve ?

$$X^q Z + X Z^q = Y^{q+1}$$

Theorem (Homma)

$p > 0$, $q = p^e \geq 3$, $K = \overline{\mathbb{F}}_p$.

$H \subset \mathbb{P}^2$: $X^qZ + XZ^q = Y^{q+1}$ Hermitian curve

$$\Delta(H) \cup \Delta'(H) = \mathbb{P}^2(\mathbb{F}_{q^2})$$

In particular, $\sharp\Delta(H) = q^3 + 1$, $\sharp\Delta'(H) = q^4 - q^3 + q^2$.

$q = 3 \Rightarrow \sharp\Delta(H) = 28$, $\sharp\Delta'(H) = 63$.

A curve having infinitely many Galois points

$q = p^e \geq 3$;

$C \subset \mathbb{P}^2$: $XZ^{q-1} - Y^q = 0$

$P = (1 : 0 : 0)$: singular point

$Q = (0 : 1 : 0)$: strange center

Distribution of Galois points (F-Hasegawa)

(1) $\Delta(C) = C \setminus \{P\}$

(2) $\Delta'(C) = \{Z = 0\} \setminus \{P, Q\}$

IV Classification Theorems

Inner Case

Theorem (F-Hasegawa)

$C \subset \mathbb{P}^2$: irred. plane curve of deg. $d \geq 4$

The Followings Are Equivalent:

(1) $\Delta(C)$: non-empty Zariski open set of C.

(2) $p > 0$, q: power of p & $C \sim x - y^q = 0$.

Outer Case

Main Theorem

$C \subset \mathbb{P}^2$: plane curve of deg. $d \geq 3$.

The **F**ollowings **A**re **E**quivalent:

(1) $\Delta'(C)$: Infinite.

(2) C: Rational & Strange with center Q;

$\exists L \subset \mathbb{P}^2$: Line s.t.

$\underline{Q \in L}$ & $\underline{L \cap \Delta'(C)\text{: Infinite}}$.

(3) $p > 0$, $C \sim$

$$\alpha_e x^{p^e} + \alpha_{e-1} x^{p^{e-1}} + \cdots + \alpha_0 x + \beta_e y^{p^e} + \beta_{e-1} y^{p^{e-1}} + \cdots + \beta_1 y^p = 0.$$

Remark.

$C \subset \mathbb{P}^2$:

$x^{p^e} + \alpha_{e-1} x^{p^{e-1}} + \cdots + \alpha_1 x^p + \alpha_0 x + \beta_e y^{p^e} + \cdots + \beta_1 y^p = 0.$

$P \in \mathrm{Sing} C$,

Q: strange center. ($P = Q$ is possible.)

Then:

(i) $\Delta(C) = \begin{cases} C \setminus \{P\} \text{ if } C \sim x - y^q = 0. \\ \emptyset \quad \text{otherwise.} \end{cases}$

(ii) $\Delta'(C) = \{Z = 0\} \setminus \{P, Q\}$.

(iii) G_R: cyclic group of order $p^e - 1$ for $\forall R \in \Delta(C)$.

(iv) $G_R \cong (\mathbb{Z}/p\mathbb{Z})^{\oplus e}$ for $\forall R \in \Delta'(C)$.