Metric Diophantine Approximation for Formal Laurent Series over Finite Fields

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Fq9, July 16th, 2009

Notation

- Field of formal Laurent series:

$$\mathbb{F}_q((T^{-1})) = \{f = a_n T^n + a_{n-1}T^{n-1} + \cdots : \ a_j \in \mathbb{F}_q, a_n \neq 0\} \cup \{0\}.$$

- Valuation induced by the general degree function:

$$|f| = q^n, \qquad |0| = 0.$$

- Analogue of $[0, 1)$:

$$\mathbb{L} = \{f \in \mathbb{F}_q((T^{-1})) : |f| < 1\}.$$

- Restricting $|\cdot|$ to $\mathbb{L}$ gives compact topological group. Denote by m the unique, translation-invariant (Haar) probability measure.

Approximation Problem - Coprime Solutions

For $f \in \mathbb{L}$ consider:

$$\left| f - \frac{P}{Q} \right| < \frac{1}{q^{n+l_n}}, \ \deg Q = n, \ Q \text{ monic}, \tag{AP}$$

where

- $P, Q \in \mathbb{F}_q[T], Q \neq 0$;
- l_n is a sequence of non-negative integers.

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Question:

For a “typical” f (with respect to m), what can be said about the number of pairs (P, Q) with $\gcd(P, Q) = 1$ solving the above Diophantine inequality?

Two Results of Inoue & Nakada

Theorem (Inoue & Nakada; 2003)

AP has either finitely or infinitely many coprime solutions for almost all f*. The latter holds iff*

$$\sum_{n} q^{n-l_n} = \infty.$$

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Theorem (Inoue & Nakada; 2003)

Let $l_n \geq n$. Then, the number of coprime solutions of AP with $n \leq N$ satisfies

$$(1-q^{-1})\Psi(N) + \mathcal{O}\left(\Psi(N)^{1/2}\left(\log \Psi(N)\right)^{3/2+\epsilon}\right) \qquad \text{a.s.},$$

where $\Psi(N) = \sum_{n \leq N} q^{n-l_n}$.

Some Notation

Assume that

$$\sum_n q^{n-l_n} = \infty \quad \text{and} \quad l_n \text{ increasing.}$$

Note that the latter implies that $l_n \geq n$ and $l_n - n$ is non-decreasing.

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Define

$$F(N) = \begin{cases} q^{-2l-2}\left(q^{l+1}(q-1) - (2l+1)(q-1)^2\right)N, & \text{if } l_n - n \to l; \\ (1-q^{-1})\Psi(N), & \text{if } l_n - n \to \infty, \end{cases}$$

where $\Psi(N) = \sum_{n \leq N} q^{n-l_n}$.

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where $\Psi(N) = \sum_{n \leq N} q^{n-l_n}$.

Finally set

$$Z_N(f) = \#\text{ coprime solutions of AP with } n \leq N.$$

CLT and LIL

Theorem (Deligero & Nakada; 2004)

As $N \to \infty$,

$$\frac{Z_N - (1-q^{-1})\Psi(N)}{\sqrt{F(N)}} \xrightarrow{d} \mathcal{N}(0,1),$$

where $\Psi(N) = \sum_{n\le N} q^{n-l_n}$.

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Theorem (Deligero, F., Nakada; 2007)

We have,

$$\limsup_{N \to \infty} \frac{|Z_N(f) - (1 - q^{-1})\Psi(N)|}{\sqrt{2F(N) \log \log F(N)}} = 1 \qquad a.s.,$$

where $\Psi(N) = \sum_{n \le N} q^{n - l_n}$.

Approximation Problem - All Solutions

For $f \in \mathbb{L}$ consider:

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where

- $P, Q \in \mathbb{F}_q[T], Q \neq 0$;
- l_n is a sequence of non-negative integers.

Question:

For a "typical" f (with respect to m), what can be said about the number of pairs (P, Q) solving the above Diophantine inequality?

A Result of Nakada & Natsui

Theorem (Nakada & Natsui; 2006)

Assume that

(i) l_n *is increasing,* $\sum_n q^{n-l_n} = \infty$*;*

(ii) *The sequence recursively defined by*

$$j_1 = \min\{n \geq 2 : l_n - l_{n-1} > 1\};$$
$$j_k = \min\{n > j_{k-1} : l_n - l_{n-1} > 1\}$$

is lacunary.

Then, the number of solutions of AP with $n \leq N$ *is asymptotic to*

$$\Psi(N) = \sum_{n \leq N} q^{n-l_n}.$$

An improved Result

Theorem (F.)

Let $l_n \geq n$. Then, the number of solutions of AP with $n \leq N$ satisfies

$$\Psi(N) + \mathcal{O}\left(\Psi(N)^{1/2} (\log \Psi(N))^{2+\epsilon}\right) \qquad \textit{a.s.},$$

where

$$\Psi(N) = \sum_{n \leq N} q^{n-l_n}.$$

Inhomogeneous Diophantine Approximation

For $f, g \in \mathbb{L}$ consider:

$$|Qf - g - P| < \frac{1}{q^{l_n}}, \ \deg Q = n, \ Q \text{ monic}, \qquad \text{(IAP)}$$

where P, Q and l_n are as before.

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Different cases:

(D) *Double metric case*: both f, g random;

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Different cases:

(D) *Double metric case*: both f, g random;

(S) *Single metric cases*:

- (S1) g fixed, f random;
- (S2) f fixed, g random.

Double Metric Case

Theorem (Ma & Su; 2008)

IAP for (D) has either finitely or infinitely many solutions for almost all (f, g). *The latter holds iff*

$$\sum_{n} q^{n-l_n} = \infty.$$

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Let $l_n \geq n$. Then, the number of solutions of IAP for (D) with $n \leq N$ satisfies

$$\Psi(N) + \mathcal{O}\left(\Psi(N)^{1/2} (\log \Psi(N))^{3/2+\epsilon}\right) \qquad a.s.,$$

where $\Psi(N) = \sum_{n \leq N} q^{n-l_n}$.

Single Metric Cases

Theorem (F.)

Let $l_n \geq n$. Then, the number of all solutions of IAP for (S1) with $n \leq N$ satisfies

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Theorem (F.)

A similar result for (S2) cannot hold.

More precisely, for any l_n there exists an f such that the number of solutions of (S2) is finite almost surely.

Restricted Diophantine Approximation

For $f, g \in \mathbb{L}$ consider:

$$|F(Q)f - g - P| < \frac{1}{q^{l_n}}, \ \deg Q = n, \ Q \text{ monic,} \qquad \text{(RAP)}$$

where P, Q, l_n are as before and F is a map from $\mathbb{F}_q[T]$ to $\mathbb{F}_q[T]$.

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Assumption and Notation:

- $\deg Q \leq \deg Q' \quad \Rightarrow \quad F(Q) \leq F(Q');$

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Assumption and Notation:

- $\deg Q \leq \deg Q' \quad \Rightarrow \quad F(Q) \leq F(Q');$
- Set
  $$\mathcal{F} = \{Q : \ Q \text{ monic and } F(Q) \neq 0\}.$$
  and
  $$\mathcal{F}_n = \{Q : Q \in \mathcal{F}, \deg Q = n\}.$$

A Theorem for Special F

Theorem (F.)

Let $l_n \geq n$ and assume that $F(Q) \in \{Q, 0\}$. Then, the number of solutions of RAP with $Q \in \mathcal{F}$ and $n \leq N$ satisfies

$$\Psi(N, \mathcal{F}) + \mathcal{O}\left(\Psi(N)^{1/2} \left(\log \Psi(N)\right)^{2+\epsilon}\right) \qquad a.s.,$$

where

$$\Psi(N) = \sum_{n \leq N} q^{n-l_n}, \qquad \Psi(N, \mathcal{F}) = \sum_{n \leq N} \#\mathcal{F}_n q^{-l_n}.$$

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Remark:

This gives a meaningful formula whenever

$$\liminf_{n\to\infty} \#\mathcal{F}_n q^{-n} > 0.$$

Consequences

Corollary

Let $l_n \geq n$ *and set* $\Psi(N) = \sum_{n \leq N} q^{n-l_n}$.

(i) *Let* $C, D \in \mathbb{F}_q[T]$ *with* $\deg C < \deg D$. *Then, the number of solutions of IAP with* $Q \equiv C\ (D)$ *and* $n \leq N$ *satisfies*

$$\frac{1}{|D|}\Psi(N) + \mathcal{O}\left((\Psi(N))^{1/2}(\log \Psi(N))^{2+\epsilon}\right) \qquad \textit{a.s.}$$

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(ii) *The number of solutions of IAP with* Q *square-free and* $n \leq N$ *satisfies*

$$(1 - q^{-1})\Psi(N) + \mathcal{O}\left((\Psi(N))^{1/2} (\log \Psi(N))^{2+\epsilon}\right) \qquad \textit{a.s.}$$

A Theorem for General F

Theorem (F.)

Let $l_n \geq n$. *Then, the number of solutions of RAP with* $Q \in \mathcal{F}$ *and* $n \leq N$ *satisfies*

$$\Psi(N, \mathcal{F}) + \mathcal{O}\left(\Psi_0(N)^{1/2} \left(\log \Psi_0(N)\right)^{3/2+\epsilon}\right) \qquad a.s.,$$

where

$$\Psi(N, \mathcal{F}) = \sum_{n \leq N} \# \mathcal{F}_n q^{-l_n}$$

and

$$\Psi_0(N) = \sum_{n \leq N} q^{-l_n} \sum_{m \leq n} \sum_{Q \in \mathcal{F}_n} \sum_{Q' \in \mathcal{F}_m} \frac{|\gcd(F(Q), F(Q'))|}{|F(Q)|}$$

Consequences

Corollary

Let $l_n \geq n$.

(i) *The number of solutions of IAP with* Q *irreducible and* $n \leq N$ *satisfies*

$$\Psi_1(N) + \mathcal{O}\left(\Psi_1(N)^{1/2}\left(\log \Psi_1(N)\right)^{3/2+\epsilon}\right) \qquad \textit{a.s.},$$

where $\Psi_1(N) = \sum_{n \leq N} n^{-1} q^{n-l_n}$.

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where $\Psi_1(N) = \sum_{n \leq N} n^{-1} q^{n-l_n}$.

(ii) *Let* $F(Q) = Q^t$ *with* $t \geq 2$. *Then, the number of solutions of RAP with* $n \leq N$ *satisfies*

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where $\Psi(N) = \sum_{n \leq N} q^{n-l_n}$.

Simultaneous Diophantine approximation

For $(f_1, \ldots, f_d) \in \mathbb{L} \times \cdots \times \mathbb{L}$ consider:

$$\left| f_j - \frac{P_j}{Q} \right| < \frac{1}{q^{n+l_n^{(j)}}},\ 1 \leq j \leq d,\ \deg Q = n,\ Q \text{ monic,} \tag{SAP}$$

where P_j, Q and $l_n^{(j)}$ are as before. Set $l_n = \sum_j l_n^{(j)}$.

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Theorem (F.)

Let $l_n \ge n$. Then, the number of all solutions of AP with $n \le N$ satisfies

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Let $l_n \geq n$. Then, the number of coprime solutions of SAP with $n \leq N$ satisfies

$$c_0 \Psi(N) + \mathcal{O}\left(\Psi(N)^{1/2+\epsilon}\right) \qquad \textit{a.s.},$$

where $\Psi(N) = \sum_{n \leq N} q^{n-l_n}$ and $c_0 > 0$ is some constant.

Thanks for Your Attention!