Classification of Rosenbloom-Tsfasman Block Codes

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Poset Codes

[Brualdi, Graves, Lawrence, 1995]

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- $\langle supp(v) \rangle =$ ideal generated by $supp(v)$

Poset Codes

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Block Codes

[Feng, Xu and Hickernell, 2006]

Notation and Definitions

- $\pi = (k_1, k_2, \ldots, k_n)$ a partition of $N \in \mathbb{N}$

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$$\omega_\pi(v) = |supp_\pi(v)|$$

- where $v = v_1 + \cdots + v_n, v_i \in V_i$, and $supp_\pi(v) = \{i; v_i \neq 0\}$

Rosenbloom-Tsfaman Block Codes

Definition

- The **Rosenbloom-Tsfasman block weight** w_π (π-**weight**) of a vector $0 \neq v = v_1 + v_2 + \ldots + v_n \in V$ is

$$w_\pi(v) = \max\{i : v_i \neq \mathbf{0}\};$$

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$$d_\pi(u, v) := w_\pi(u - v).$$

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- The π-**Rosenbloom-Tsfasman space** (or simply a π-**space**): (V, d_π)

Minimal Weight

- Definition
  The π **-minimal distance** of a linear code $C \subseteq V$:

$$d_{\pi} = d_{\pi}\left(C\right) := \min\left\{d_{\pi}\left(c, c'\right) : c \neq c' \in C\right\}$$

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  The π-**minimal weight** of C:

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Generalized Weights - Wei

- Definition

  The **generalized Rosenbloom-Tsfasman block weight** (or π-**weight**) of $D \subseteq V$:

$$\|D\| := \max\{w_\pi(x) : x \in D\};$$

Generalized Weights - Wei

- Definition
  The **generalized Rosenbloom-Tsfasman block weight** (or π-**weight**) of $D \subseteq V$:
  $$\|D\| := \max\{w_\pi(x) : x \in D\};$$

- Definition
  The r-**th Rosenbloom-Tsfasman block weight** (or r-**th** π-**weight**) of a linear code $C \subseteq V$:
  $$d_r = d_r(C) := \min\{\|D\| : D \subseteq C, \dim(D) = r\}$$

Generalized Weights - Wei

- Definition
  The π-**generalized weight hierarchy** $(d_1, d_2, \ldots, d_k)$ of an $[N; k; d_1, \ldots, d_k]$ **linear code**.

Generalized Weights - Wei

- Definition
  The π-**generalized weight hierarchy** $(d_1, d_2, \ldots, d_k)$ of an $[N; k; d_1, \ldots, d_k]$ **linear code**.
  - **Remark:** The generalized weight hierarchy is increasing but not necessarily strict.

Contents

Best looking generating matrix

BEST LOOKING GENERATING MATRIX

$$\begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \dots & (\mathrm{Id}_{s_m \times s_m} | \mathbf{0}) \\ 0 & \cdots & 0 & 0 & \cdots & (\mathrm{Id}_{s_{m-1} \times s_{m-1}} | \mathbf{0}) & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & (\mathrm{Id}_{s_1 \times s_1} | \mathbf{0}) & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix}$$

(Kind of) Triangular Generating Matrix

Theorem (Extension of Ozen and Siap - 2004)

Let C be an $[N;k]$ linear code. Then C admits a generating matrix of the form

$$\begin{pmatrix} G_{s_m\pi_1} & \cdots & G_{s_m\pi_{t_1}} & G_{s_m\pi_{t_1+1}} & \cdots & G_{s_m\pi_{t_{m-1}}} & G_{s_m\pi_{t_{m-1}+1}} & \cdots \\ G_{s_{m-1}\pi_1} & \cdots & G_{s_{m-1}\pi_{t_1}} & G_{s_{m-1}\pi_{t_1+1}} & \cdots & G_{s_{m-1}\pi_{t_{m-1}}} & \mathbf{0} & \cdots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ G_{s_1\pi_1} & \cdots & G_{s_1\pi_{t_1}} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots \end{pmatrix}$$

where G_{st} is an $s \times t$ matrix with $s \leq t$, for each $1 \leq i \leq m$ the rank of $G_{s_i\pi_{t_i}}$ is s_i and $s_1 + s_2 + \ldots + s_m = k$.

- Definition
  A matrix like the one in Theorem 7 is called a matrix of **type** $((s_1, t_1), \ldots, (s_m, t_m))$.

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- Definition
  A matrix of type $((s_1, t_1), \ldots, (s_m, t_m))$ is said to be in a **canonical form** if $G_{s_i \pi_j} = \mathbf{0}$ for $1 \leq j \leq t_i - 1$ and $G_{s_i \pi_{t_i}} = \left(\mathrm{Id}_{s_i \times s_i} | \mathbf{0}_{s_i \times \left(\pi_{t_i} - s_i\right)}\right)$ for every $i = 1, \ldots, m$.

Best looking form generating matrix = Canonical Form

$$\begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \ldots & (\mathrm{Id}_{s_m \times s_m} | \mathbf{0}) \\ 0 & \cdots & 0 & 0 & \cdots & (\mathrm{Id}_{s_{m-1} \times s_{m-1}} | \mathbf{0}) & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & (\mathrm{Id}_{s_1 \times s_1} | \mathbf{0}) & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix}$$

- Definition
  A **linear isometry** T of the π-metric space (V, d_π) is a linear transformation $T : V \longrightarrow V$ that preserves π-metric:$d_\pi(T(u), T(v)) = d_\pi(u, v)$ for every $u, v \in V$.

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- Definition
  Two linear codes $C, C' \subseteq V$ are considered to be **equivalent** if there is a linear isometry $T : V \rightarrow V$ such that $T(C) = C'$.

Theorem (Canonical Form)

Let C be $[N; k; d_1, \ldots, d_k]$ linear code with generating matrix G of type $((s_1, t_1), \ldots, (s_m, t_m))$. Then there is a linear isometry T of (V, d_π) such that the linear code $T(C)$ has a generating matrix in a canonical form of type $((s_1, t_1), \ldots, (s_m, t_m))$.

Proof: We start with a generating matrix $G = (g_{ij})$ of type $((s_1, t_1), \ldots, (s_m, t_m))$ (existence ensured by previous Theorem).

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- Consider the π-decomposition

$$v_i = v_{i1} + \ldots + v_{in},$$

with $v_{ij} \in V_j$

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- Define

$$w_i = v_{it_l} = v_i - p_{(1,2,\ldots,t_l-1)}\left(v_i\right).$$

with $p_{(1,2,\ldots,t_l-1)}$ projection in the coordinate space.
Notation:

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with $p_{(1,2,\ldots,t_l-1)}$ projection in the coordinate space.
Notation:

- $$v_\iota^i = v_{s_1+\ldots+s_{\iota-1}+i} \text{ and } w_\iota^i = w_{s_1+\ldots+s_{\iota-1}+i}$$

- $$\alpha_\iota = \{v_\iota^1, v_\iota^2, \ldots, v_\iota^{s_\iota}\} \text{ and } \beta_\iota = \{w_\iota^1, w_\iota^2, \ldots, w_\iota^{s_\iota}\}$$

  are both linearly independent.

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  are both linearly independent.
- Let $\widetilde{\beta}_\iota = \left\{u_\iota^1, \ldots, u_\iota^{\pi_{t_\iota} - s_\iota}\right\}$ be such that $\beta_\iota \cup \widetilde{\beta}_\iota$ is a base for V_{t_ι}.

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- It follows that $\alpha_\iota \cup \widetilde{\beta}_\iota$ is also a base for V_{t_ι}.
- $$\beta = \bigcup_{\iota=1}^{m} \left(\beta_\iota \cup \widetilde{\beta}_\iota\right) \text{ and } \alpha = \bigcup_{\iota=1}^{m} \left(\alpha_\iota \cup \widetilde{\beta}_\iota\right)$$
  are bases of $V_{t_1} \oplus V_{t_2} \oplus \ldots \oplus V_{t_m}$.

- Let $\Lambda = \{t_1, \ldots, t_m\}$, $\gamma_\iota = \left\{e_\iota^1, \ldots, e_\iota^{\pi_\iota}\right\}$ is the canonical base for the block space V_ι and

$$\gamma = \bigcup_{\iota \notin \Lambda} \gamma_\iota,$$

we conclude that

$$\alpha \cup \gamma \text{ and } \beta \cup \gamma$$

are bases of V.

- $L : V \to V$ the L.T. defined by

$$L\left(v_\iota^i\right) = w_\iota^i,\ L\left(u_\iota^i\right) = u_\iota^i \text{ and } L\left(e_\iota^i\right) = e_\iota^i \text{ if } \iota \notin \Lambda.$$

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  and

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- $S : V \to V$ the L.T. defined by

$$S\left(w_\iota^i\right) = e_{t_\iota}^i,\ S\left(u_\iota^j\right) = e_{t_\iota}^{s_\iota + j} \text{ and } S\left(e_\iota^i\right) = e_\iota^i \text{ if } \iota \notin \Lambda.$$

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- $S(L(G))$ is a matrix in canonical form.
- [-, Muniz, Panek, 2007 ensures both L and S are isometries.

Classification Theorem

Theorem (Classification Theorem)

The canonical form of generating matrices classifies π-codes, in the sense that any equivalence class of codes contains a unique code that has a generating matrix in canonical form.

Proof.

The existence of a code that has a generating matrix in canonical form was proved in the Canonical Form Theorem. The uniqueness of such a code follows from the fact that different matrices in canonical form generate codes with different π-weight hierarchy. □

Contents

Generalized Singleton Bound

Theorem (Generalized Singleton Bound)

Let C be an $[N;k]$ linear code of type $((s_1,t_1),\dots,(s_m,t_m))$. Then for each $1 \leq j \leq m$,

$$d_{s_1+s_2+\ldots+s_j}(C) \leq n - m + j$$

In particular, we have the **Singleton Bound**

$$d_1(C) \leq n - m + 1.$$

Spectrum

Theorem

Let C be a code of type $((s_1, t_1), \ldots, (s_m, t_m))$. If $j = t_i$ for some i, then

$$A_j^{(r)} = \sum_{s=1}^{\min\{s_i, r\}} \frac{(q)_{\sigma_{i-1}}(q)_{s_i}}{(q)_{r-s}(q)_{\sigma_{i-1}-r+s}(q)_s(q)_{s_i-s}q^{(r-s)s(\sigma_{i-1}-r-s)(s_i-s)}}$$

and $A_j^{(r)} = 0$ otherwise.

Packing Radius

Definition

The **packing radius** $R = R(C)$ of a linear code C is

$$R := \max\left\{r : B_\pi(c; r) \cap B_\pi\left(c'; r\right) = \varnothing \text{ for every } c \neq c' \in C\right\}.$$

Theorem

The packing radius of an $[N; k; d_\pi]$ *linear* π*-code is*

$$R(C) = d_\pi(C) - 1.$$

MDS Codes

Theorem

Let (V, d_π) be the π-space with $\pi_i = 1$ for every $1 \leq i \leq n$ and let C be an $[N; k]$ π-code. Then C is MDS iff C is π-perfect.

Covering Radius

Theorem

An $[N; k; d_1]$ π-code C is quasi-perfect iff

$$p_{(d_1+1,\ldots,n)}(C) = V_{d_1+1} \oplus V_{d_1+2} \oplus \ldots \oplus V_n$$

and $p_{d_1}(C) \neq V_{d_1}$.

Syndromes and Decoding

Theorem

Let C be a π-code and $R(C) = d_\pi(C) - 1$ its packing radius. If $u \in V$ is a π-coset leader of a coset C_v and $w_\pi(u) \leq R(C)$, then u is the unique π-coset leader of C_v.

- M.M.S. Alves, L. Panek and M. Firer, **Error-block codes and poset metrics**, Advances in Mathematics of Communications 2 (2008) 95-111.
- R.A. Brualdi, J.S. Graves and K.M. Lawrence, **Codes with a poset metric**, Discrete Mathematics 147 (1995) 57-72.
- K. Feng, L. Xu and F.J. Hickernell, **Linear error-block codes**, Finite Fields and Their Applications 12 (2006) 638-652.
- S. Ling and F. Özbudak, **Constructions and bounds on linear error-block codes**, Designs, Codes and Cryptography (2007), doi:10.1007/s10623-007-9119-9.
- M. Ozen and I. Siap, **On the estructure and decoding of linear codes with respect to Rosenbloom-Tsfasman metric**, Selçuk Journal of Applied Mathematics 5 (2) (2004) 25-31.
- L. Panek, M. Firer, H.K. Kim and J.Y. Hyun, **Groups of**