New caps in $PG(k, 5)$

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Definition: Caps

A n-cap in $PG(k,q)$ is a set of n points in $PG(k,q)$ no three of which are collinear.
A cap C is called affine If there is a hyperplane containing no points of C.

A n-cap in $PG(k,q)$ is equivalent to a linear q-ary code

$$[n, n-k-1, 4]_q$$

What is the maximum size of a cap in $PG(k, q)$?

Known exact values of $m_2(k, q)$:

- $m_2(k, 2) = 2^k$
- $m_2(2, q) = q + 1$ for q odd.
- $m_2(2, q) = q + 2$ for q even.
- $m_2(3, q) = q^2 + 1$
- $m_2(4, 3) = 20, \quad m_2(5, 3) = 56, \quad m_2(4, 4) = 41.$

What is the maximum size of a cap in $PG(k, q)$?

Lower bounds (constructions) on $m_2(4, q)$, q odd:

Theorem (B. E. 2000)

Let q be an odd prime-power. Then there exist in $PG(4, q)$ caps of the following cardinality:

$$\begin{array}{ll} (5q^2 - 2q - 7)/2 & \text{if } q \equiv 1 \pmod 8, \\ (5q^2 - 8q - 13)/2 & \text{if } 3 < q \equiv 3 \pmod 8, \\ (5q^2 - 6q - 11)/2 & \text{if } q \equiv 5 \pmod 8, \\ (5q^2 - 4q - 9)/2 & \text{if } q \equiv 7 \pmod 8. \end{array}$$

Theorem (B. E. 2001)

There exists a 66-cap in $PG(4, 5)$.

What is the maximum size of a cap in $PG(k, q)$?

Lower bounds (constructions) on $m_2(5, q)$, q odd:

Theorem (B. E. 1999; Kroll, Vincenti 2008)

There is a $\{(q+1)(q^2+3)\}-$cap in $PG(5, q)$.

Theorem (B. E. 2001)

There exists a 186-cap in $PG(5, 5)$.

A new cap in $PG(5,5)$

Theorem (B. E.)

There exists a 195-cap in $PG(5,5)$.

Consequences

Theorem (B. E. 1999)

Assume the following exist:

- An n-cap $K_1 \subset PG(k, q)$ possessing a tangent hyperplane, and
- an m-cap $K_2 \subset PG(l, q)$ possessing a tangent hyperplane.

Then there is an $\{nm - 1\}$-cap in $PG(k + l, q)$.

The 26-cap in $PG(3, 5)$, the 66-cap in $PG(4, 5)$ and the 195-cap in $PG(5, 5)$ each possess a tangent hyperplane.

Corollary (B. E.)

There exists an

- 5069-cap in $PG(8, 5)$
- 130951-cap in $PG(11, 5)$ (130951=(26*26-1)*194+1)

Lower bounds on $m_2(k, 5)$

$k\backslash q$	5	
2	**6**	
3	**26**	
4	66	
5	195	186
6	675	
7	1715	
8	5069	4700
9	17124	
10	43876	
11	130951	120740

Notation

Definition

The parity of a nonzero element of $\mathbb{F}_q$ is its quadratic remainder symbol.

Definition

With slight abuse of notation we say that $B \subset \mathbb{F}_q^k$ is an affine cap, if any nontrivial affine linear combination of at most three vectors of B is non-zero.

Definition

Let $A \subset \mathbb{F}_q^k$, $x \in \mathbb{F}_q^k$ and $U \subset \mathbb{F}_q^l$ then we define:

$$(A, U) := \{(a, u) \in \mathbb{F}_q^{k+l} | a \in A, u \in U\}$$

$$(x, U) := \{(x, u) \in \mathbb{F}_q^{k+l} | u \in U\}$$

Construction

Theorem

Let $A, B \subset \mathbb{F}_q^k$ *such that A is a cap in PG(k-1,q) and B is an affine cap.*
Let $U, V \subset \mathbb{F}_q^l$ *such that U is a cap in PG(l-1,q) and V is an affine cap.*
Furthermore let .... "condition C" be fulfilled
Then

$$C := (A, V) \cup (B, U) \subset PG(k + l - 1, q)$$

is a $(|A| \cdot |V| + |B| \cdot |U|)$*-cap.*

The ”condition C”

(... some technical conditions ....) and for

- any vanishing linear combination of two distinct vectors of A and one vector of B, the coefficients of the two vectors in A have different parity.
- any vanishing linear combination of two vectors of B, which are no multiples of each other, and one vector of A, the coefficients of the two vectors in B have different parity.

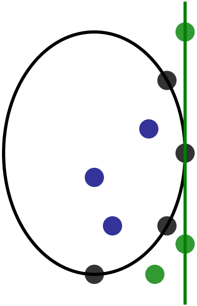
- any vanishing linear combination of two distinct vectors of U and one vector of V, the coefficients of the two vectors in U have the same parity.
- any vanishing linear combination of two vectors of V, which are no multiples of each other, and one vector of U, the coefficients of the two vectors in V have the same parity

The Barlotti arc

The arc is the conic section $\mathcal{Q} = V(Y^2 - XZ)$ are the points $(x : y : z) \in PG(2, q)$ satisfying $y^2 - xz = 0$.
$\mathcal{Q}$ is a $q + 1$-cap in $PG(2, q)$.

The exterior points are the points on the tangents of $\mathcal{Q}$, for these:

$y^2 - xz$ is a non-zero square.



The remaining points are called the interior points of $\mathcal{Q}$, for these:

The pair A, B

Lemma

Let $q = 5$, A the following representatives of the points the arc $\mathcal{Q}$:

$$A := \{(1, y, y^2) | y \in \mathbb{F}_q\} \cup \{(0, 0, 1)\}$$

and B the following two representatives for each interior point of $\mathcal{Q}$:

$$B := \{(x, xv, xv^2 + 2/x) | x \in \mathbb{F}_q^*, v \in \mathbb{F}_q\}$$

This pair A, B has the needed properties for the construction.

The pair U, V for $l = 3$

Lemma

Let $\mathcal{Q}' = V((X-Z)^2 - 2Y(X+Z))$.
For $q = 5$ the points of $\mathcal{Q}'$ are exterior points of $\mathcal{Q}$ and vice versa.

Lemma

Let $q = 5$, and U be the following set of represenatives of the arc $\mathcal{Q}$:

$$U := \{(1, y, y^2) | y \in \mathbb{F}_q\} \cup \{(0:0:1)\}$$

Let W be the following set of represenatives of $\mathcal{Q}'$:

$$W = \{(1,0,1),(0,1,0),(0,1,2),(2,1,0),(1,1,2),(2,1,1)\}$$

Let $V := W \cup -W$. The pair U, V has the needed properties for the construction.

The pair U, V for $l = 2$

Lemma

Let $q = 5$,

$$U := \{(1,0),(0,1)\}$$

$$V := \{(1,1),(-1,1),(1,-1),(-1,-1)\}$$

This pair U, V *has the needed properties for the construction.*

Extension points

For $q = 5$, $l = 2$ the construction leads to a cap with $(q-1)(q+1) + 2q(q-1) = 64$ points in $PG(4,5)$.

This cap is extendable by the two points

$$(0 : 0 : 0 : 1 : \pm 2)$$

For $q = 5$, $l = 3$ the construction leads to a cap with $2(q+1)^2 + (q+1)q(q-1) = 192$ points in $PG(5,5)$.

This cap is extendable by the three points of the form

$$(0 : 0 : 0 : 1 : * : *)$$