Parameterization of algebraic tori

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joint work with Reynald Lercier

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Outline

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Why use algebraic tori ?

Discrete-log based cryptography needs a convenient group structure.

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$$\begin{aligned} \#\mathbb{F}_{q^n}^{\times} &= \prod_{d|n} \Phi_d(q) \\ &= \Phi_1(q) \dots \underbrace{\Phi_n(q)}_{\#T_n}. \end{aligned}$$

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T_n : subgroup of $\mathbb{F}_{q^n}^{\times}$ of *"optimal"* order.

Example

$$\#\mathbb{F}_{q^6}^{\times} = \underbrace{(q-1)}_{T_1 = \mathbb{F}_q^{\times}} \underbrace{(q+1)}_{T_2 \subset \mathbb{F}_{q^2}^{\times}} \underbrace{(q^2+q+1)}_{T_3 \subset \mathbb{F}_{q^3}^{\times}} \underbrace{(q^2-q+1)}_{T_6}.$$

Also an algebraic variety

Definition

We define the *algebraic torus* by the following relation :

$$T_n(\mathbb{F}_q) = \left\{ \alpha \in \mathbb{F}_{q^n}^\times : \alpha^{\Phi_n(q)} = 1 \right\}.$$

It is equivalent to write that the algebraic torus $T_n(\mathbb{F}_q)$ is the intersection of the kernels of the norms $N_{\mathbb{F}_{q^n}/F}$, over all $F \subsetneq \mathbb{F}_{q^n}$.

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Remark

The dimension of $T_n(\mathbb{F}_q)$ is $\varphi(n) = \deg(\Phi_n)$.

In brief

Two ways of considering the algebraic torus T_n.

$T_n \subset \mathbb{F}_{q^n}^\times$,	$T_n(\mathbb{F}_q) = \left\{\alpha \in \mathbb{F}_{q^n}^\times : \alpha^{\Phi_n(q)} = 1\right\}$,
$\# T_n = \Phi_n(q)$.	$\dim T_n = \varphi(n)$.

Remark.

$$\#\mathbb{F}_{q^n}^\times = \prod_{d|n} \Phi_d(q),$$

$$\Rightarrow \quad \boxed{\mathbb{F}_{q^n}^\times = \prod_{d|n} T_d(\mathbb{F}_q)}$$

Outline

Compact representation

We want to send an element of the torus.
Naive representation : $T_n \subset \mathbb{F}_{q^n} \quad \longrightarrow \quad n$ coordinates.
Clever representation : $dim(T_n) = \varphi(n) \quad \longrightarrow \quad \varphi(n)$ coordinates.

Compact representation

We want to send an element of the torus.
Naive representation : $T_6 \subset \mathbb{F}_{q^6} \longrightarrow$ 6 coordinates.
Clever representation : $dim(T_6) = \varphi(6) \longrightarrow$ 2 coordinates.

Example

Parameterization of T_6 with $\varphi(6) = 2$ coordinates.

- XTR (Lenstra, Verheul)
- CEILIDH (Silverberg, Rubin)

Idea for a bijection (van Dijk, Woodruff)

Idea of the bijection $\Theta : \; T_n \times \Pi \longrightarrow \tilde{\Pi}$.
Recall that $\mathbb{F}_{q^n}^{\times} = \prod_{d|n} T_d$. Example in the case $n = 15$.

$$T_{15} \longrightarrow \mathbb{F}_{q^{15}}^{\times}$$

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$T_{15} \xrightarrow{\Theta}$

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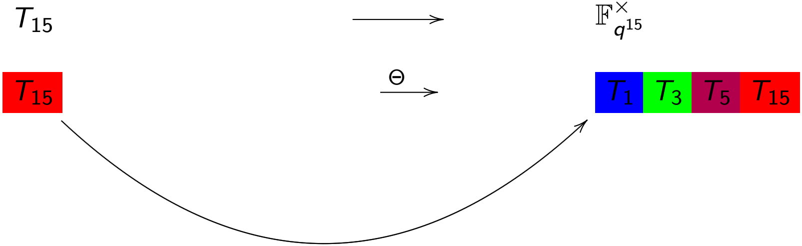

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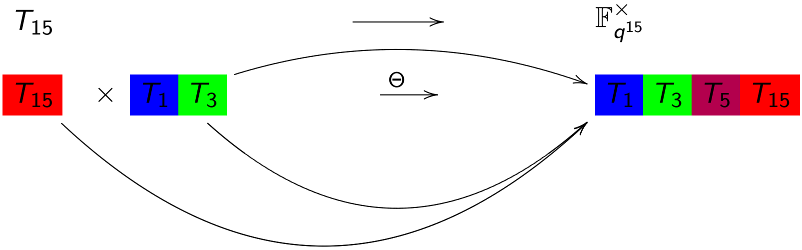

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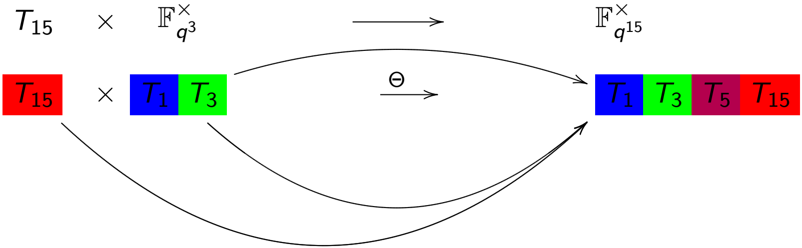

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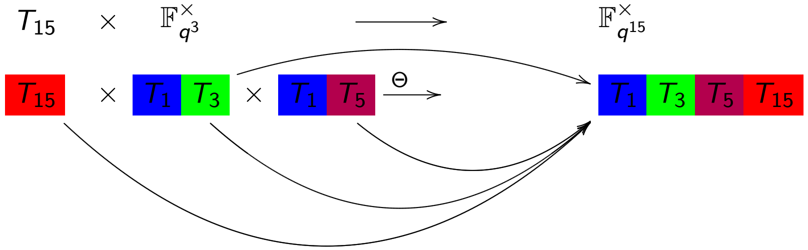

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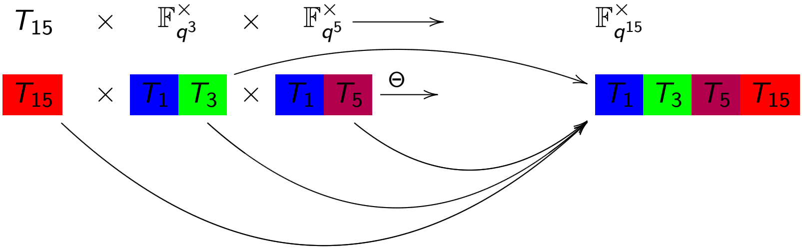

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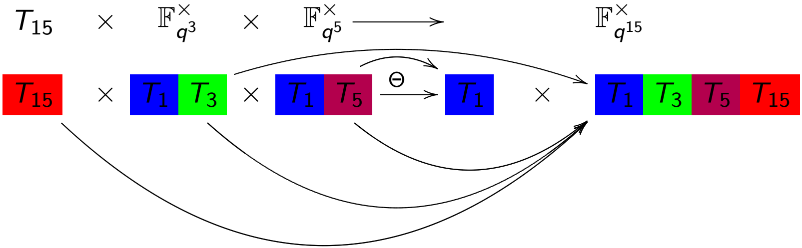

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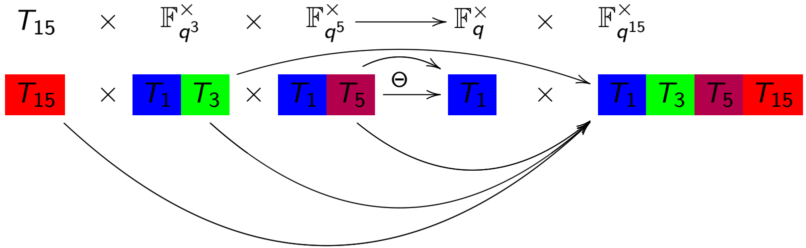


General formula

μ denotes the Möbius function.

$$\Theta : T_n(\mathbb{F}_q) \times \prod_{\substack{d|n \\ \mu(n/d)=-1}} \mathbb{F}_{q^d}^\times \longrightarrow \prod_{\substack{d|n \\ \mu(n/d)=+1}} \mathbb{F}_{q^d}^\times$$

The bijection step by step

$$
\begin{array}{ccccc}
x & & x_3 & & x_5 \\
T_{15} & \times & \mathbb{F}_{q^3}^{\times} & \times & \mathbb{F}_{q^5}^{\times}
\end{array}
$$

The bijection step by step

$$
\begin{array}{ccccc}
x & & x_3 & & x_5 \\
T_{15} & \times & \mathbb{F}_{q^3}^{\times} & \times & \mathbb{F}_{q^5}^{\times} \\
\downarrow & & \downdownarrows \ (1) & & \downdownarrows \\
T_{15} & & (T_1 \times T_3) & & (T_1 \times T_5)
\end{array}
$$

The bijection step by step

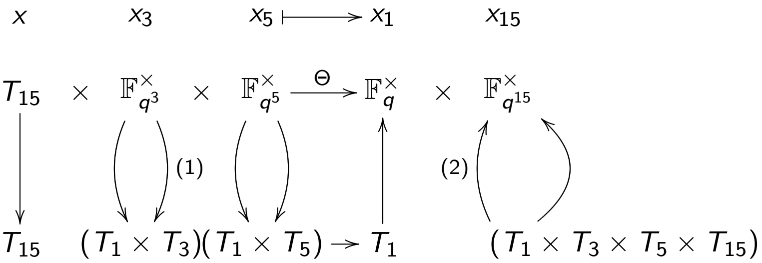
$$x \qquad x_3 \qquad x_5$$

$$T_{15} \times \mathbb{F}_{q^3}^{\times} \times \mathbb{F}_{q^5}^{\times}$$

$$\downarrow \qquad \downarrow\downarrow \ (1) \qquad \downarrow\downarrow$$

$$T_{15} \quad (T_1 \times T_3)(T_1 \times T_5) \to T_1 \qquad (T_1 \times T_3 \times T_5 \times T_{15})$$

The bijection step by step



Outline

First operation

It is the decomposition of a point of $\mathbb{F}_{q^d}^\times$ into components in the tori T_1 and T_d.
We perform this operation in the most natural way :

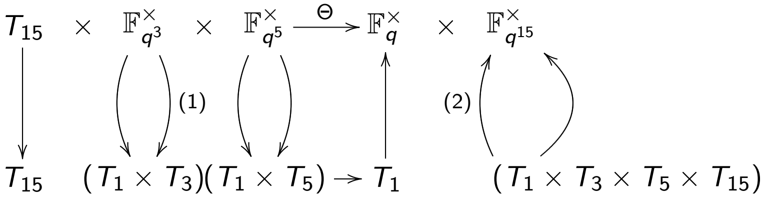
$$\begin{array}{ccccc} \mathbb{F}_{q^d}^\times & \xrightarrow{(1)} & T_1 & \times & T_d \\ x_d & \longmapsto & \Big(x_d^{\Phi_d(q)} & , & x_d^{\Phi_1(q)} \Big) \end{array}$$

Cost

$\Phi_d(q)$ has degree $\varphi(d)$ in q.
So performing exponentiations to the power $\Phi_d(q)$ requires

$$O(\varphi(d) \log q) \text{ multiplications in } \mathbb{F}_q.$$

The bijection step by step



Second operation

Let's see on the example $n = 15$: It is a recombination of coordinates in the tori T_1, T_3, T_5, T_{15}.

$$\begin{array}{ccc} T_1 \times T_3 \times T_5 \times T_{15} & \longrightarrow & \mathbb{F}_{q^{15}}^{\times} \\ (x_1, x_3, x_5, x_{15}) & \xrightarrow{(2)} & x \end{array}$$

$$x = x_1^{w_1} x_3^{w_3} x_5^{w_5} x_{15}^{w_{15}} \quad \text{where} \sum_{d|15} \frac{q^{15}-1}{\Phi_d(q)} w_d = 1.$$

Cost

For each divisor d of 15, we have $w_d \leqslant q^{15} - 1$. So exponentiation requires

$$O(15 \log q) \text{ multiplications.}$$

Outline

Elliptic normal bases (Couveignes, Lercier)

Theorem

For any extension $\mathbb{F}_{q^n}$ of $\mathbb{F}_q$, there exists a normal basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ such that both

- the product of two elements
- exponentiation of an element to a power of q

can be performed in linear time.

The construction of such bases uses elliptic curves, hence the name of *elliptic bases*.

First operation

Recall the first decomposition :

$$\begin{array}{cccccc} \mathbb{F}_{q^d}^\times & \xrightarrow{(1)} & T_1 & \times & T_d \\ x_d & \longmapsto & \Big(x_d^{\Phi_d(q)} & , & x_d^{\Phi_1(q)} \Big) \end{array}$$

Cost

$\Phi_1(q) = q - 1$, $\Phi_3(q) = q^2 + q + 1$ and $\Phi_5(q) = q^4 + q^3 + q^2 + q + 1$.
So performing exponentiations to these powers requires

$$O(\varphi(d)) \text{ multiplications in } \mathbb{F}_q.$$

Second operation

Recall the recombination on the right hand side :

$$\begin{array}{ccc} T_1 \times T_3 \times T_5 \times T_{15} & \longrightarrow & \mathbb{F}_{q^{15}}^{\times} \\ (x_1, x_3, x_5, x_{15}) & \xrightarrow{(2)} & x \end{array}$$

$$x = x_1^{w_1} x_3^{w_3} x_5^{w_5} x_{15}^{w_{15}} \quad \text{where} \sum_{d|15} \frac{q^{15}-1}{\Phi_d(q)} w_d = 1.$$

Cost

w_d are actually polynomials in q. So once again we have exponentiations to a power of q and a certain number of multiplications, depending on the coefficients. In practice, the polynomials w_d have very convenient coefficients.

Modular inverses of cyclotomic polynomials

Theorem

For all p and r distinct prime numbers,

(i) $\Phi_p^{-1} \bmod \Phi_1 = 1/p$ and
$\Phi_1^{-1} \bmod \Phi_p = (-1/p)(X^{p-2} + 2X^{p-3} + \ldots + p - 1)$.

(ii) $\Phi_{pr}^{-1} \bmod \Phi_1 = 1$ and $\Phi_1^{-1} \bmod \Phi_{pr} = \sum_{i=0}^{\varphi(pr)-1} v_i X^i$ with $v_i \in \{-1, 0, 1\}$.

(iii) $\Phi_{pr}^{-1} \bmod \Phi_p = \frac{1}{r} \sum_{i=0}^{d} X^i$ with $d \equiv r - 1 \bmod p$ and
$\Phi_p^{-1} \bmod \Phi_{pr} = \frac{1}{r} \sum_{i=0}^{\varphi(pr)-1} v_i X^i$ with $v_i < r$.

(iv) $\Phi_p^{-1} \bmod \Phi_r = \sum_{i=0}^{\varphi(r)-1} v_i X^i$ with $v_i \in \{0, -1, +1\}$.

Results

Global cost of the communication :

- Before improvement : $O(n^3 \log^2 q)$ operations.
- After improvement : $O(P(n) \log q)$ operations.

Usual cost of a classical D.H. cryptosystem : $O(n^2 \log^2 q)$.

Outline

Things to do (among others)

- Case of n product of more than two primes.
- Question of a *birational* parameterization.

Thank you for your attention

and

enjoy you meal !

Compression and cryptography

Idea of the bijection $\Theta : \; T_n \times \Pi \longrightarrow \tilde{\Pi}$.

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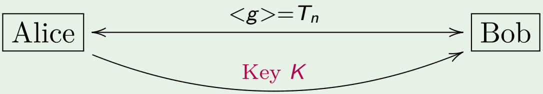
Diffie-Hellman multiple key agreement



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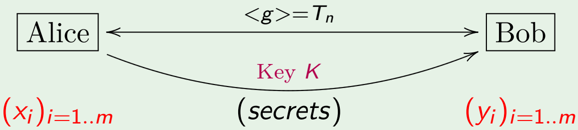

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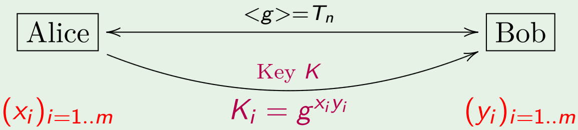

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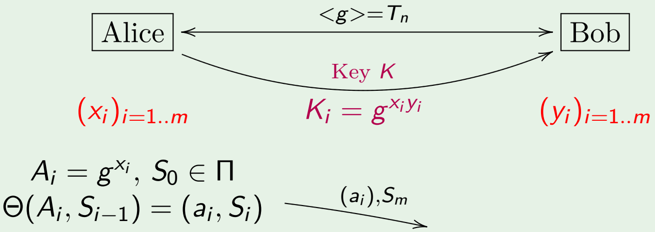

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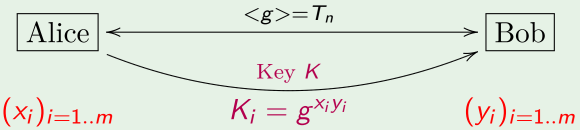

$$A_i = g^{x_i},\; S_0 \in \Pi$$
$$\Theta(A_i, S_{i-1}) = (a_i, S_i)$$

$\xrightarrow{(a_i), S_m}$

$$\Theta^{-1}(a_m, S_m) = (A_m, S_{m-1})$$
$$\Theta^{-1}(a_{m-1}, S_{m-1}) = (A_{m-1}, S_{m-2})$$

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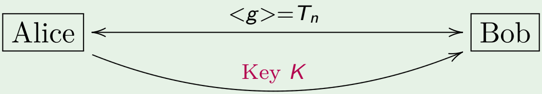

$(x_i)_{i=1..m}$ $\quad$ $K_i = g^{x_i y_i}$ $\quad$ $(y_i)_{i=1..m}$

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