On $\mathcal{C}$-ultrahomogeneous graphs and digraphs

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The study of ultrahomogeneous graphs (resp. digraphs) is traceable back to the work of Sheehan 1974, Gardiner 1976, Ronse 1978, (resp. Fraisse 1954, Lachlan 1980, Cherlin 1988).

In the present work, some explicit $\mathcal{C}$-ultrahomogeneous graphs and digraphs are obtained, via the line of research on $\mathcal{C}$-ultrahomogeneous graphs conceived in 2007 by D. Isaksen et al., as follows.

Given a collection $\mathcal{C}$ of (di)graphs closed under isomorphisms, a (di)graph G is $\mathcal{C}$-*ultrahomogeneous*, or $\mathcal{C}$-UH, if every isomorphism between two induced members of $\mathcal{C}$ in G extends to an automorphism of G.

If $\mathcal{C} = \{H\}$, we say that such a G is $\{H\}$-UH or H-UH.

In the work of Isaksen et al., $\mathcal{C}$-UH graphs are defined and studied when $\mathcal{C}$ is the collection of either **(a)** complete graphs, or **(b)** disjoint unions of complete graphs, or **(c)** complements of those unions.

First, let us present a $\{K_4, K_{2,2,2}\}$-UH graph G_3^1 that fastens objects of (a) and (c), namely K_4 and $K_{2,2,2}$.

The *Fano plane* $\mathcal{F} = P(2,2)$ is the *binary projective plane* (= space of lines of the field GF_2^3), e.g. a (7_3)-configuration with points

$$1, 2, 3, 4, 5, 6, 7$$

and *Fano lines*

$$123, 145, 167, 246, 257, 347, 356.$$

The map Φ that sends the points $1, 2, 3, 4, 5, 6, 7$ respectively onto the lines 123, 145, 167, 246, 257, 347, 356 has the following *duality properties*:

(1) each point p of $\mathcal{F}$ pertains to the lines $\Phi(q)$, with $q \in \Phi(p)$;

(2) each Fano line ℓ contains the points $\Phi(k)$, where k runs over the lines passing through $\Phi(\ell)$.

Given a point p of $\mathcal{F}$, the collection of lines through p is called a *pencil* of $\mathcal{F}$.

A linearly ordered presentation of these lines is an
ordered pencil through p.

An ordered pencil v through p, denoted $v = (p, q_a r_a, q_b r_b, q_c r_c)$, is orderly composed, in reality, by the lines

$$pq_a r_a, pq_b r_b, pq_c r_c.$$

There are $3! = 6$ ordered pencils through any point p of $\mathcal{F}$.

The claimed graph G_3^1 is presented in a paper just appeared in the Australasian J. of Combinatorics, (June 2009).

Ordered pencils constitute the vertex set of G_3^1, with any two vertices

$$v = (p, q_a r_a, q_b r_b, q_c r_c)$$

and

$$v' = (p', q'_a r'_a, q'_b r'_b, q'_c r'_c)$$

adjacent whenever the following two conditions hold:
(1) $p \neq p'$;
(2) $|p_i r_i \cap p'_i r'_i| = 1$, for $i = a, b, c$.

The 3 points of intersection resulting from item (2) form an *ordered Fano line* by taking into account the subindex order

$$a < b < c.$$

An alternate definition of G_3^1 can be given via Φ^{-1}, with the vertices of G_3^1 seen as the ordered Fano lines $x_a x_b x_c$, with any two such vertices adjacent if their associated Fano lines share the entry in $\mathcal{F}$ of exactly one of its 3 positions, either a or b or c.

Example of an edge in G_3^1 and its image via ϕ^{-1}:

Ordered pencils $\leftrightarrow$ ordered lines

$$(1,23,45,67) \leftrightarrow 123$$
$$(2,13,46,57) \leftrightarrow 145$$

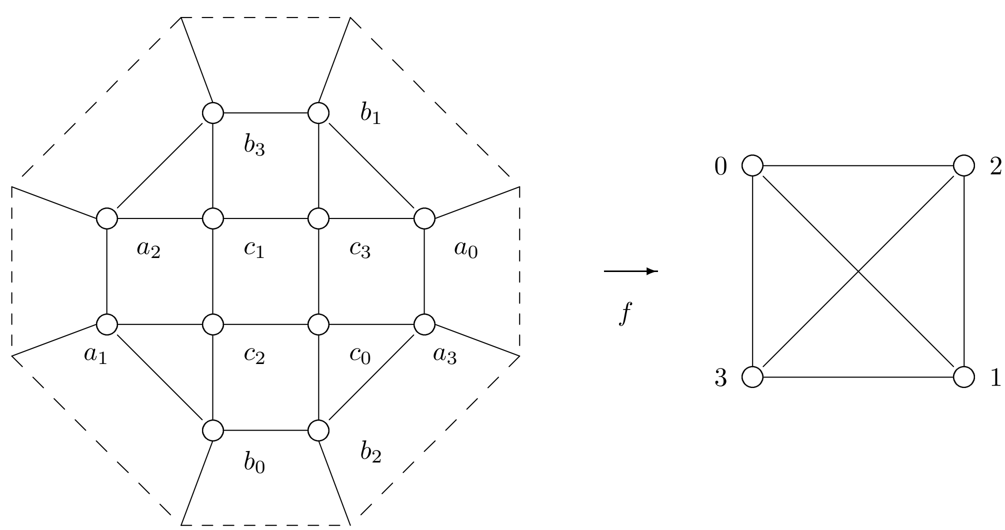
Theorem 1 *The graph* G_3^1 *is a 12-regular* K_2*-fastened* $\{K_4,\ K_{2,2,2}\}$*-UH graph of order 42 and diameter 3. Each vertex of* G_3^1 *is incident to exactly 3 copies of* $K_{2,2,2}$ *and 4 copies of* K_4*. Moreover,* G_3^1 *is the Menger graph of a self-dual* (42_4)*-configuration.*

That G_3^1 is K_2-fastened means that each edge of G_3^1 is the intersection of exactly one copy of K_4 and exactly one copy of $K_{2,2,2}$ in G_3^1.

It is important to note that G_3^1 is not the line graph of any graph. (Menger graphs of self-dual configurations are generally line graphs.)

Idea of proof:

The 12 neighbors of any vertex v of G_3^1 induce the open neighborhood of v in G_3^1, namely a *hemi-rhombicuboctahedron* (obtained from the *rhombicuboctahedron* by identification of antipodal vertices and edges):



The corresponding closed neighborhood of v in G^1_3 looks like the center figure here:

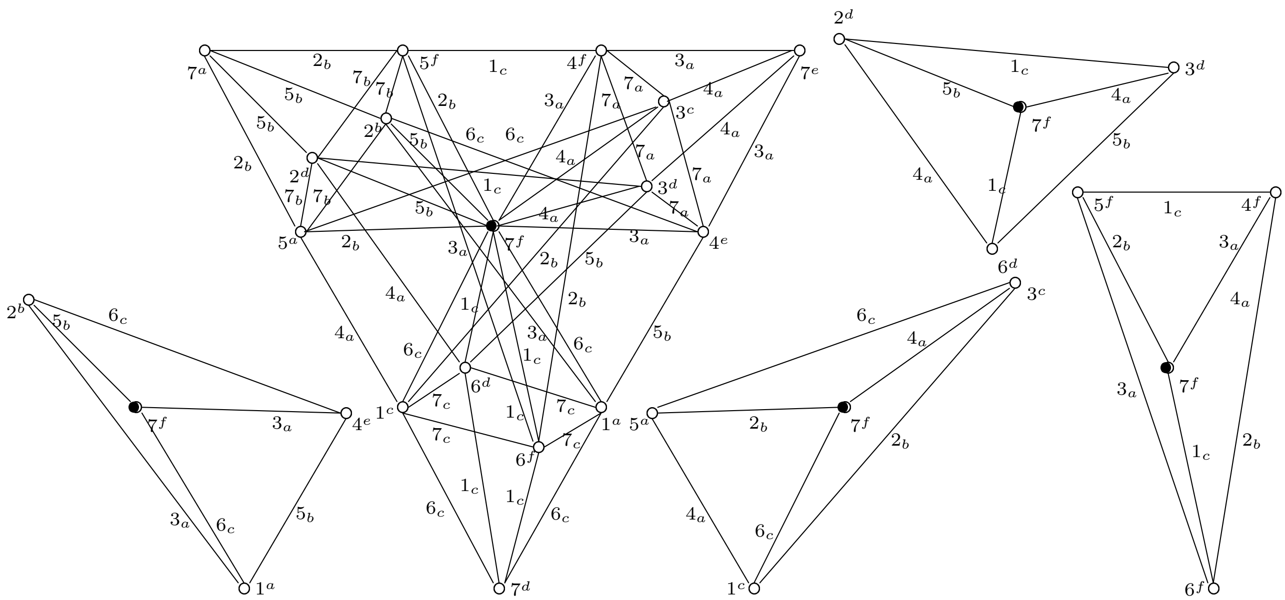


Figure 1: Disposition of copies of $K_{2,2,2}$ and K_4 at vertex 7^f in G

Next:
On how cubic distance transitive graphs yield $\mathcal{C}$-UH graphs

In addition to the Menger graph of the self-dual (42_4)-configuration above, which is a K_2-fastened $\{K_4, K_{2,2,2}\}$-UH graph, we produced recently the Menger graph of a self-dual (102_4)-configuration, which is a K_3-fastened $\{K_4, L(Q_3)\}$-UH graph.

The procedure that yields such an object starts by taking an undirected graph G as a digraph and by considering each edge of G as a pair of oppositely oriented or OO arcs.

Let M be a sub(di)graph of a (di)graph H and let G be both an M-UH and an H-UH (di)graph.

G is a (fastened) $(H;M)$-*UH (di)graph* if given a copy H_0 of H in G containing a copy M_0 of M, there exists exactly one copy $H_1 \neq H_0$ of H in G with

$$V(H_0) \cap V(H_1) = V(M_0)$$

and

$$A(H_0) \cap \bar{A}(H_1) = A(M_0),$$

where $\bar{A}(H_1)$ is formed by those arcs whose orientation is reversed with respect to the arcs of $A(H_1)$, and moreover: no more vertices or edges than those in M_0 are shared by H_0 and H_1.

In the undirected case, the vertex and edge conditions above are condensed as $H_0 \cap H_1 = M_0$. This can be generalized by saying that a graph G is an *ℓ-fastened* $(H;M)$-UH graph if given a copy H_0 of H in G containing a copy M_0 of M, then there exist exactly ℓ copies $H_i \neq H_0$ of H in G such that $H_i \cap H_0 = M_0$, for $i = 1, 2, \ldots, \ell$, and such that no more vertices or edges than those in M_0 are shared by each two of $H_0, H_1, \ldots, H_\ell$.

Theorem 2 *Let G be a cubic distance transitive graph (CDT) of arc transitivity k and girth g. Then, G is a $(C_g; P_k)$-UH graph and has exactly $2^{k-2}3ng^{-1}$ g-cycles.*

Theorem 3 *The CDT graphs G of girth g and arc transitivity k that are not $(\vec{C}_g;\vec{P}_k)$-UH digraphs are the Petersen graph, the Heawood graph and the Foster graph. The remaining nine CDT graphs here are fastened $(\vec{C}_g;\vec{P}_k)$-UH.*

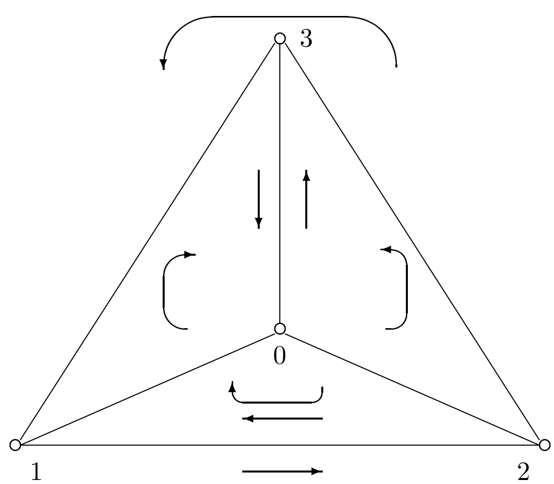
Given a $(\vec{C}_g;\vec{P}_k)$-UH graph G, an assignment of an orientation to each g-cycle of G, such that the two g-cycles shared by each $(k-1)$-path have opposite orientations, yields a $(\vec{C}_g;\vec{P}_k)$-*orientation assignment* or $(\vec{C}_g;\vec{P}_k)$-*OA*.
The collection of η oriented g-cycles corresponding to the η g-cycles of G, for a particular $(\vec{C}_g;\vec{P}_k)$-OA, will be called an $(\eta\vec{C}_g;\vec{P}_k)$-*OAC*.

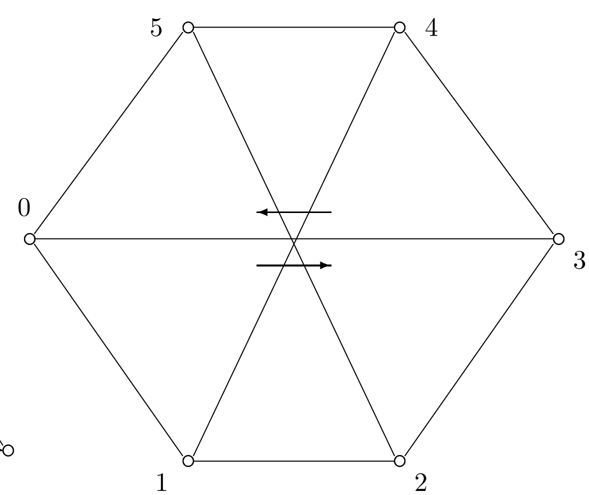
The graph $G = K_4$ on vertex set $\{1, 2, 3, 0\}$ admits the $(4\vec{C}_3; \vec{P}_2)$-OAC:

$$\{(123), (210), (301), (032)\}.$$

The graph $G = K_{3,3}$ obtained from K_6 (with vertex set $\{1, 2, 3, 4, 5, 0\}$) by deleting the edges of the triangles $(1, 3, 5)$ and $(2, 4, 0)$ admits the $(9\vec{C}_4; \vec{P}_3)$-OAC:

$$\{(1234), (3210), (4325), (1430), (2145),$$
$$(0125), (5230), (0345), (5410)\}.$$





Let G be either the *Pappus, Desargues, Coxeter* or *Biggs-Smith* graph. Consider the collection $\mathcal{C}_g^{k-1}(G)$ of $(k-1)$-powers of the oriented g-cycles of a $(\eta\vec{C}_g;\vec{P}_k)$-OAC of G.

If $k=3$, then each arc $\vec{e}$ of a member C^2 of $\mathcal{C}_g^2(G)$ is indicated by the middle vertex of the 2-arc $\vec{E}$ in C for which $\vec{e}$ stands, while the tail and head of $\vec{e}$ are indicated by the tail and head of $\vec{E}$, respectively.

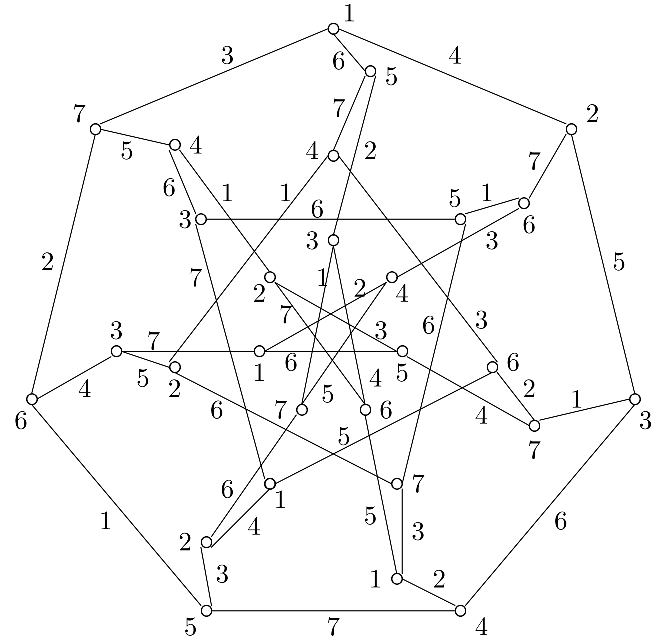
If $k=4$, which is the case of the Biggs-Smith graph, cube powers C^3 in $\mathcal{C}_g^3(G)$ are considered. But such C^3 are formed by 3 3-cycles. Such 3-cycles arrange themselves into 102 copies of K_4.

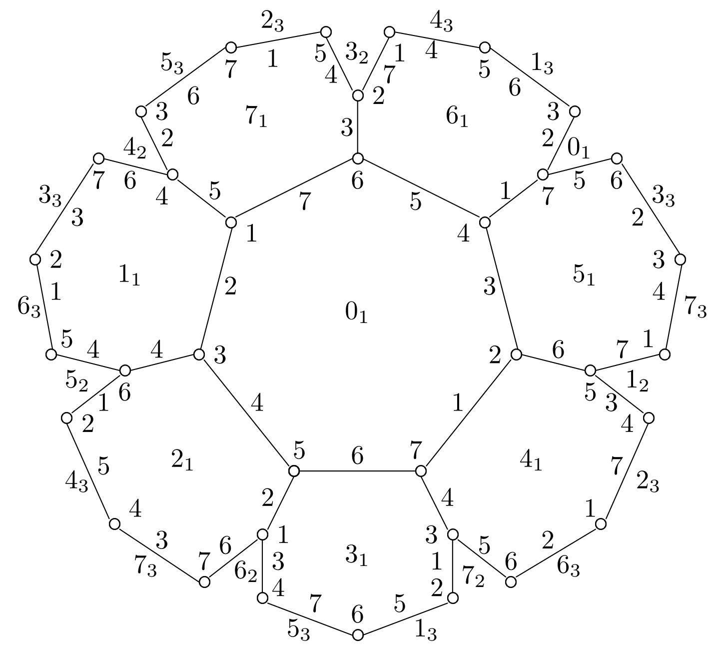
The Fano plane $\mathcal{F}$, with point set $J_7 = \{1, \ldots, 7\}$ and line set $\{124, 235, 346, 457, 561, 672, 713\}$, bestows a coloring to the vertices and edges of the *Coxeter graph* $G = Cox$.

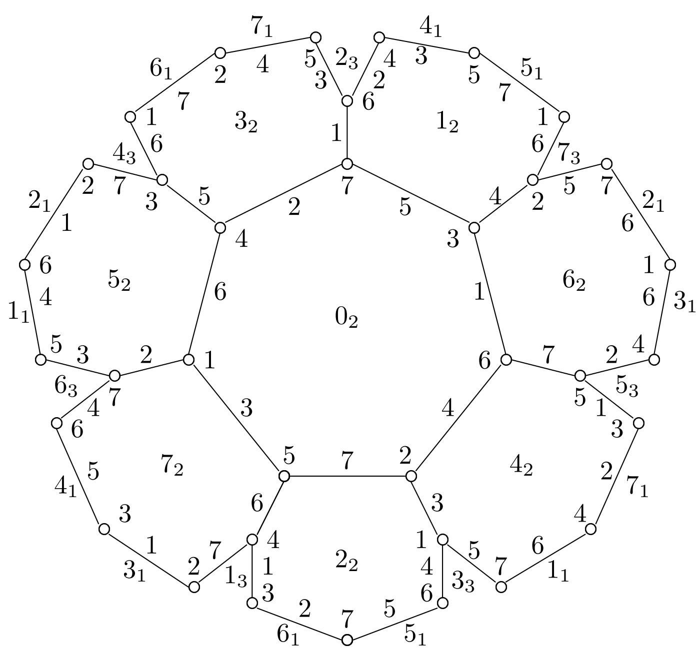
The colors of each vertex v of G and of its three incident edges form a quadruple q whose complement $\mathcal{F} \setminus q$ is a triangle of $\mathcal{F}$ used as a 'customary' vertex denomination for v. Then:

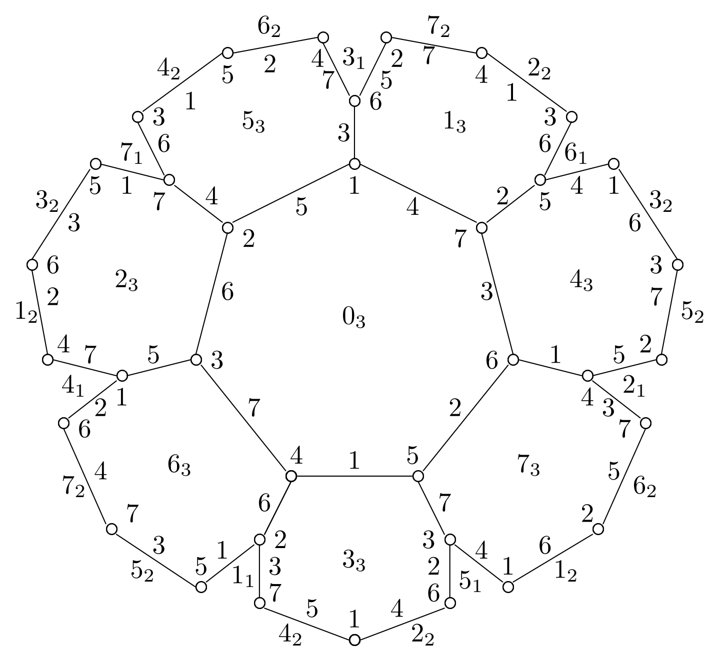
(a) the triple formed by the colors of the edges incident to each vertex of G is a line of $\mathcal{F}$;

(b) the color of each edge e of G together with the colors of the end-vertices of e form a line of $\mathcal{F}$.









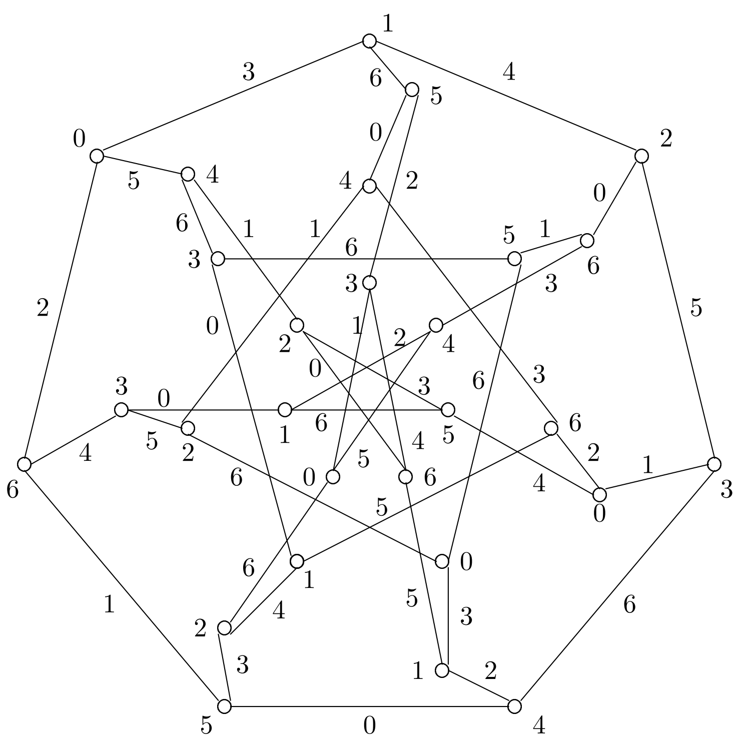
Theorem 4 *The Klein graph*

$$Y(Cox)$$

on 56 vertices is a $(C_7; P_2)$*-UH graph composed by* 24 7*-cycles that yield the Klein map* $\{7,3\}_8$ *in the 3-torus* T_3*.* $\square$

A $\vec{C}_4$-UH digraph which is strongly connected and without OO arcs:

The Fano plane $\mathcal{F}$ is taken with point set $J_7 = \{0, 1, \ldots, 6\}$ and line set $\{124,\ 235,\ 346,\ 450,\ 561,\ 602,\ 013\}$. The vertices and edges of the Coxeter graph $G = Cox$ are colored as follows:



which suggests that each vertex v of Cox can be considered as a *pencil of ordered lines* of $\mathcal{F}$:

$$xb_1c_1,\ \ xb_2c_2,\ \ xb_0c_0, \qquad (1)$$

corresponding to the three edges e_1, e_2, e_0 incident to v, respectively, and denoted by $[x, b_1c_1, b_2c_2, b_0c_0]$, where x is the color of v in the figure and b_i and c_i are the color of e_i and the color of the endvertex of e_i other than v, for $i \in \{1, 2, 0\}$.

Moreover, two such vertices $[x, b_1c_1, b_2c_2, b_0c_0], [x', b'_1c'_1, b'_2c'_2, b'_0c'_0]$ are adjacent in Cox if $b_ic_i \cap b'_ic'_i$ is constituted by just one element d_i, for one $i \in \{1, 2, 0\}$, and the resulting triple $d_1d_2d_0$ is a line of $\mathcal{F}$.

In this definition of Cox, there is not any ordering imposed on the lines of each pencil representing a vertex of Cox.

Consider the digraph D whose vertices are the *ordered pencils of ordered lines* of $\mathcal{F}$ as in (1) above. Each such vertex will be denoted as $(x, b_1c_1, b_2c_2, b_0c_0)$, where $b_1b_2b_0$ is a line of $\mathcal{F}$. An arc between two vertices of D, say from

$$(x,\ b_1c_1,\ b_2c_2,\ b_0c_0) \text{ to } (x',\ b'_1c'_1,\ b'_2c'_2,\ b'_0c'_0),$$

is established if and only if

$$x = c'_i,\ b'_{i+1} = c_{i-1},\ b'_{i-1} = c_{i+1},$$
$$x' = c_i,\ c'_{i+1} = b_{i+1},\ c'_{i-1} = b_{i-1},$$
$$b'_i = b_i,$$

for some, $i \in \{1, 2, 0\}$.

This way, we obtain oriented 4-cycles in D, such as

$$((0,26,54,31),(6,20,15,43),(0,26,31,54),(6.20.43.15)).$$

A simplified notation for the vertices (x,yz,uv,pq) of D is yup_x. With such a notation, the adjacency sub-list of D departing from the vertices of the form yup_0 is (with rows indicated a,b,c,d,e,f):

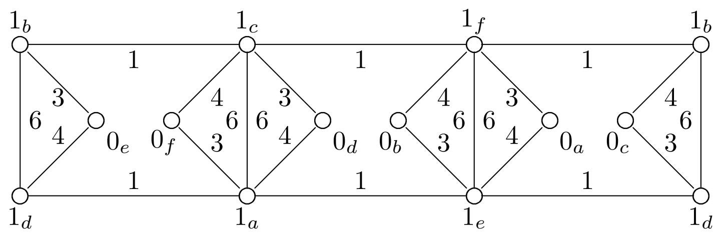
$124_0:165_3,325_6,364_5;$	$235_0:214_6,634_1,615_6;$	$346_0:352_1,142_5,156_2;$	$156_0:142_3,352_4,346_2;$
$142_0:156_3,346_5,352_6;$	$253_0:241_6,651_4,643_6;$	$364_0:325_1,165_2,124_5;$	$165_0:124_3,364_2,325_4;$
$214_0:235_6,615_3,634_5;$	$325_0:364_1,124_6,165_1;$	$436_0:412_5,532_1,516_2;$	$516_0:532_4,412_3,436_2;$
$241_0:253_6,643_5,651_3;$	$352_0:346_1,156_4,142_1;$	$463_0:421_5,561_2,523_1;$	$561_0:523_4,463_2,421_3;$
$412_0:436_5,516_3,532_6;$	$523_0:561_4,421_6,463_4;$	$634_0:615_2,235_1,214_5;$	$615_0:634_2,214_3,235_4;$
$421_0:463_5,523_6,561_3;$	$532_0:516_4,436_1,412_4;$	$643_0:651_2,241_5,253_1;$	$651_0:643_2,253_4,241_3.$

From this sub-list, the adjacency list of D, for its $168 = 24 \times 7$ vertices, is obtained via translations mod 7. Let us represent each vertex yup_0 of D by means of a symbol i_j, where $j = a,b,c,d,e,f$ stands for the successive rows of the table above and $i \in \{0,1,2,4\}$.

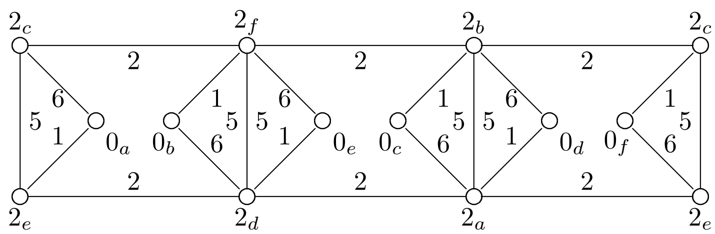
These symbols i_j are assigned to the lines yup avoiding $0 \in \mathcal{F}$, and thus to the yup_0, as follows:

i_j	$j{=}a$	$j{=}b$	$j{=}c$	$j{=}d$	$j{=}e$	$j{=}f$
$i{=}0$	124_0	142_0	214_0	241_0	412_0	421_0
$i{=}1$	235_0	253_0	325_0	352_0	523_0	532_0
$i{=}2$	346_0	364_0	436_0	463_0	634_0	643_0
$i{=}4$	156_0	165_0	516_0	561_0	615_0	651_0

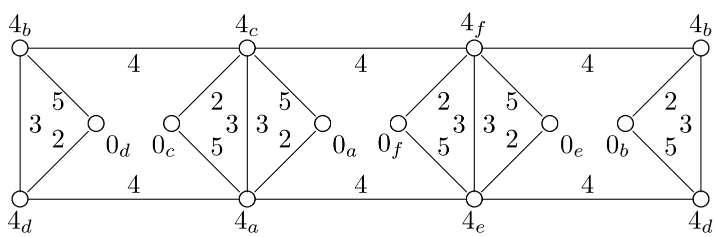
The quotient graph $D/\mathbf{Z}_7$ admits a split representation::



1_b 1_c 1_f 1_b
1 1 1
3 4 3 4 3 4
6 6 6 6 6 6
4 3 4 3 4 3
0_e 0_f 0_d 0_b 0_a 0_c
1 1 1
1_d 1_a 1_e 1_d



2_c 2_f 2_b 2_c
2 2 2
6 1 6 1 6 1
5 5 5 5 5 5
1 6 1 6 1 6
0_a 0_b 0_e 0_c 0_d 0_f
2 2 2
2_e 2_d 2_a 2_e



4_b 4_c 4_f 4_b
4 4 4
5 2 5 2 5 2
3 3 3 3 3 3
2 5 2 5 2 5
0_d 0_c 0_a 0_f 0_e 0_b
4 4 4
4_d 4_a 4_e 4_d

in which:

(a)the 18 oriented 4-cycles shown are interpreted all with counterclockwise orientation;

(b)the three vertices indicated by 0_j, for each $j \in \{a, \ldots, f\}$, represent just one vertex of D/Z_7, so they must be identified;

(c)the leftmost arc in each one of the three connected graphs must be identified with the corresponding rightmost arc by parallel translation;

(d) the arcs are indicated with voltages mod 7 whose additions with the corresponding tail symbols $\in J_7$ yield the corresponding head symbols.

All the oriented 4-cycles of D are obtained by uniform translations mod 7 from these 18 oriented 4-cycles; thus, there are just $126 = 7 \times 18$ oriented 4-cycles of D.

Our construction of D shows that the following statement holds.

Theorem 5 *The digraph D is a strongly connected $\vec{C}_4$-UH digraph on 168 vertices, 126 pairwise disjoint oriented 4-cycles, with regular indegree and outdegree both equal to 3 and no circuits of lengths 2 and 3.* $\square$

Extension of the ordered pencil technique for G_3^1

A graph G is said to be $\vec{\mathcal{C}}$-*homogeneous* if for each two isomorphic induced subgraphs $X_1, X_2 \in \mathcal{C}$ in G and arcs v_1w_1, v_2w_2 of X_1, X_2, resp., there exists an isomorphism

$$f : X_1 \to X_2,$$

with $f(v_1) = v_2$ and $f(w_1) = w_2$, extending to an automorphism of G.

If $\mathcal{C}$ is the minimal class containing two nonisomorphic graphs X_1 and X_2, then a $\vec{\mathcal{C}}$-homogeneous graph is said to be $\{X_1, X_2\}$-*homogeneous*.

For each $(r,\sigma) \in \mathbf{Z}^2$ with $r > 2$ and $\sigma \in (0, r-1)$, we introduce a connected
$\{K_{2s}, T_{ts,t}\}$-homogeneous graph G_r^σ
that is not $\{K_{2s}, T_{ts,t}\}$-UH
for $r > 3$, where:
K_{2s} is the complete graph on $2s$ vertices and
$T_{ts,t}$ is the t-partite Turán graph on s vertices per part (a total of ts vertices) with:
$t = 2^{\sigma+1} - 1$ and $s = 2^{r-\sigma-1}$, but $r \leq 8$ and $\rho = r - \sigma \leq 5$.

To obtain these objects, we need to work with the
projective geometry $P(r-1, 2)$
(=space of lines of the field GF_2^r).

Let A_0 be a $(\sigma - 1)$-subspace of $P(r-1)$. The collection of all the σ-subspaces of $P(r-1)$ containing A_0 is called the (r,σ)-*pencil* of $P(r-1)$ through A_0.
A linearly ordered presentation of this pencil is said to be an
(r,σ)-*ordered pencil* of $P(r-1)$ through A_0.

An (r,σ)-ordered pencil v of
$P(r-1)$ through A_0 has the form

$$v = (A_0 \cup A_1, \ldots, A_0 \cup A_{m_1}),$$

where A_1, $\ldots$, A_{m_1} are the nontrivial cosets of GF_2^r mod its subspace $A_0 \cup \{\bar{0}\}$, with
$m = 2^{r-\sigma} - 1$.
A shorthand for v:

$$v = (A_0, A_1, \ldots, A_m).$$

In order to keep notation, the empty set of $P(r-1)$ is said to be a $(-i)$-*subspace* of $P(r-1)$, for every negative integer i. The (r,σ)-ordered pencils of $P(r-1)$ constitute the set of vertices $v = (A_0, A_1, \dots, A_m)$ of a graph $\mathcal{G}_r^\sigma$, with an edge between each two vertices $v = (A_0, A_1 \dots, A_m)$ and $v' = (A'_0, A'_1 \dots, A'_m)$ such that:

1. $A_0 \cap A'_0$ is a $(\sigma-2)$-subspace of $P(r-1)$;

2. for each $1 \le i \le m$, $A_i \cap A'_i$ is a nontrivial coset of F_2^r mod $(A_0 \cap A'_0) \cup \{\bar{0}\}$;

3. $U(v,v') = \cup_{i=1}^m (A_i \cap A'_i)$ is an $(r-2)$-subspace of $P(r-1)$; (needed only if $(r,\sigma) \neq (3,1)$).

Let v_r^σ be the lexicographically smallest (r,σ)-ordered pencil in $\mathcal{G}_r^\sigma$. and let u_r^σ be its lexicographically smallest neighbor in $\mathcal{G}_r^\sigma$. Examples:

v_3^1=(1,23,45,67), u_3^1=(2,13,46,57), $(U(v_3^1,u_3^1)$=347);

v_4^1=(1,23,45,67,89,ab,cd,ef), u_4^1=(2,13,46,57,8a,9b,ce,df), $(U(v_4^1,u_4^1)$=3479bcf);

v_4^2=(123,4567,89ab,$cdef$), u_4^2=(145,2367,89cd,$abef$, $(U_(v_4^2,u_4^2)$=16789ef).

Then, the component of $\mathcal{G}_r^\sigma$ containing v_r^σ is the connected $\{K_{2s},T_{ts,t}\}$-homogeneous graph G_r^σ that we claimed above.