Problem sheet 6

1. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 3 & 7 & 4 & 6 & 1 & 2 \end{pmatrix}^{-1} (3\ 7\ 2) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 2 & 3 & 4 & 1 & 5 & 7 \end{pmatrix}$. As a product of disjoint cycles, it is equal to $(1\ 6\ 5)$ so it has order 3, and it is even (it is equal to $(1\ 5)(1\ 6)$).

2. $(x, y) \sim (x, y)$: yes, take $\alpha = 1$.
$(x, y) \sim (a, b) \Rightarrow (a, b) \sim (x, y)$: Since $(x, y) \sim (a, b)$ we have $\alpha \cdot (x, y) = (a, b)$. Therefore $(a, b) = \alpha^{-1} \cdot (x, y)$ i.e., $(a, b) \sim (x, y)$ (since $\alpha^{-1} > 0$).
$((x, y) \sim (a, b)$ and $(a, b) \sim (c, d)) \Rightarrow (x, y) \sim (c, d)$: Yes: We have $\alpha \cdot (x, y) = (a, b)$ and $\beta \cdot (a, b) = (c, d)$ for some $\alpha, \beta > 0$. Then $\alpha\beta \cdot (x, y) = (c, d)$ i.e., $(x, y) \sim (c, d)$ (since $\alpha\beta > 0$).

   For the equivalence classes:

   $$[(x, y)] = \{(a, b) \mid (a, b) = \alpha \cdot (x, y) \text{ for some } \alpha > 0\}.$$

   So $[(x, y)]$ is the set of points that you obtain out of (x, y) by multiplying it by a positive real:

   - If $(x, y) \neq (0, 0)$ then $[(x, y)]$ is the half-line starting at $(0, 0)$ (but not containing it) and containing (x, y).
   - If $(x, y) = (0, 0)$ then $[(x, y)]$ is $\{(0, 0)\}$.

3. (a) We have $(xy)x = x(yx)$ and by hypothesis $xy = yx$.

   (b) We have $(xy)^2 = e$ i.e., $xyxy = e$. Multiplying on both sides on the left by x we get $x^2yxy = x$, so $yxy = x$ (since $x^2 = e$). Multiplying on both sides on the right by y we get $yxy^2 = xy$, so $yx = xy$ (since $y^2 = e$).

4. Let us draw the triangle point up, with A the point on top, B the bottom-right point, and C the bottom-left one. A symmetry f of the triangle must send A to either A, B or C. We consider all three cases:

   - $f(A) = A$. Then we have two possibilities: $f(B) = B$, in which case we must have $f(C) = C$, so $f = \text{id}$. The other one is $f(B) = C$, so $f(C) = B$ and f is the symmetry across the line going through A and the middle of $[B, C]$.

- $f(A) = B$. The two possibilities are $f(B) = C$ and $f(C) = A$ (so f is the rotation around the centre of the triangle and angle $2\pi/3$) and $f(B) = A$ and $f(C) = C$, so f is the symmetry acorss the line going through C and the middle of $[A, B]$.
- $f(A) = C$. Similar.

Writing the multiplication table (also called the Cayley table) is left to you.

5. We have to show an equivalence, so we need to show that each side of it implies the other. In other words: we proceed in 2 steps, each time assuming that one side holds, then out of this proving the other.

   Assume $x \sim y$. Then $[x] = [y]$. $f(x) = i$ means $x \in A_i$, so $[x] = A_i$. Similarly $f(y) = j$ means $y \in A_j$, i.e. $[y] = A_j$. Since $[x] = [y]$ we get $A_i = A_j$ so $f(x) = f(y)$.

   Assume $f(x) = f(y)$, in other words, $i = f(x) = f(y)$ for some i. We then have $x, y \in A_i$. Since A_i is an equivalence class (and equivalence classes with non-empty intersection are equal) we get $A_i = [x]$ and $A_i = [y]$ so $[x] = [y]$. In particular $y \in [x]$ and by definition of $[x]$: $x \sim y$.