Problem sheet 5

1. (a) The permutation is equal to (1 2)(1 5)(1 4)(1 7)(2 6)(2 7)(2 4), which is odd.

   (b) The permutation is equal to (1 3 7)(2 6) = (1 7)(1 3)(2 6), which is odd.

   You can observe that a cycle of odd length is even, and a cycle of even length is odd.

2. Let σ be an even permutation, so $\sigma = \tau_1 \cdots \tau_{2k}$ where $\tau_1, \ldots, \tau_{2k}$ are transpositions different from the identity. We know that $\sigma^{-1} = \tau_{2k}^{-1} \cdots \tau_1^{-1}$, and that $\tau_i^{-1} = \tau_i$. Therefore $\sigma^{-1} = \tau_{2k}^{-1} \cdots \tau_1^{-1}$ is even.

   Let σ be an odd permutation and γ be an even permutation. Then $\sigma = \tau_1 \cdots \tau_{2n+1}$, $\gamma = \pi_1 \cdots \pi_{2k}$, where the τ_i and π_i are transpositions. Then $\sigma\tau = \tau_1 \cdots \tau_{2n+1}\pi_1 \cdots \pi_{2k}$, and the number of transpositions appearing in this product is odd.

3. $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (1\ 2\ 3)$ is a cycle of odd length so is an even permutation (a cycle of odd length is always even, a cycle of even length is always odd, see the observation before Theorem 3.27). If we want σ to be a cycle of length 4, σ will be odd, and the left-hand side of the equation will be odd, so cannot be equal to τ if τ is even.

4. (a) Let S be the set of all UCD students and B the relation "having the same birthday", i.e. if x and y are students, xBy means that x and y have the same birthday. Show that B is an equivalence relation on S.

   Reflexive: x has always the same birthday as x, so xBx.
   Symmetric: If xBy, i.e. x has the same birthday as y, then y has the same birthday as x, i.e. yBx.
   Transitive: If xBy and yBz, i.e. x has the same birthday as y and y has the same birthday as z, then x has the same birthday as z, i.e. xBz.

   (b) Let $f : A \to B$ be a function. We define a relation R on A by

$$xRy \Leftrightarrow f(x) = f(y).$$

Show that R is an equivalence relation on A.

The proof is the same as the previous one, just replace each occurence of "u has the same birthday as v" by $f(u) = f(v)$.

5. We define a relation on $\mathbb{Z}$ by: xRy if and only if x and y have a common divisor greater than 1. Show that R is not an equivalence relation.

   It is not reflexive: 1 is not in relation with 1 (since no divisor of 1 is greater than 1).
   It is enough to show that it is not an equivalence relation, but we could also check that it is not transitive ($2R6$ and $6R3$, but $2R3$ is false). It is symmetric though.