Problem sheet 11

1. If H is a subgroup of G, we know that $|H|$ is 1 or p (since it divides $|G|$). If $|H| = p$ then we must have $H = G$ (H is included in G and they have the same finite number of elements). If $|H| = 1$, then we must have $H = \{e\}$ (we know that H contains e, and it has only one element).

2. A_n contains the identity, so is not empty. If α and β are two even permutations, we have that α can be written as a product of r transpositions and β can be written as a product of s transpositions with r and s even. Then $\alpha\beta$ can be written as a product of $r+s$ transpositions, and $r + s$ is even, so $\alpha\beta \in A_n$.

   If $\alpha \in A_n$, then $\alpha = \tau_1 \cdots \tau_r$ with the τ_i transpositions and r even. Then $\alpha^{-1} = \tau_r^{-1} \cdots \tau_1^{-1} = \tau_r \cdots \tau_1$ (recall that $\tau_i^{-1} = \tau_i$ since they are transpositions). Therefore $\alpha^{-1} \in A_n$.

3. Let $x \in H \cap K$. Since $x \in H$ (and H is a group) we know that $|x|$ divides $|H|$. Similarly, since $x \in K$ we know that $|x|$ divides $|K|$. Since $\gcd(|H|, |K|) = 1$ we get that $|x| = 1$, i.e., $x = e$.

4. (a) K is a non-empty subset of S_3, is closed under taking the product of any two of its elements, and is closed under taking the inverse of any one of its elements (you need to do the computations to check all this, but it is not much work). So K is a subgroup of S_3. Other method: You can observe that K is the set of all possible powers of $(1\ 2\ 3)$, i.e. K is the subgroup generated by $(1\ 2\ 3)$, so is a subgroup.

   The order of K is 3, which is a prime number, so K is cyclic (seen in class).

   (b) We saw in the proof of Lagrange's theorem (as pointed out after the proof), that the number of different left cosets of K is $|S_3|/|K| = 2$.

   One of them is clearly $\mathrm{id}\, K = K$. For the other, you can try to compute some σK at random until you find a result different from K, but you can also use exercise 4 from problem sheet 10: To get $\sigma K \neq K$ you need to take σ not in K. We take for instance $\sigma = (1\ 2)$. Then $\sigma K = K(1\ 2) = \{(1\ 2), (1\ 2)(1\ 2\ 3), (1\ 2)(1\ 3\ 2)\}$. I leave the two computations to you.

   All the left cosets of K will be either K or σK (since we know that there are only two of them).

(c) Since the number of left cosets of a subgroup H is $|S_3|/|H|$, we want a subgroup with two elements. The subgroup generated by an element of order 2 will have two elements, so we can take $H = \langle(1\ 2)\rangle$, the subgroup generated by $(1\ 2)$, so $H = \{\mathrm{id}, (1\ 2)\}$.

5. We know from analysis that the map exp is bijective, so we only need to check the other part of the definition of isomorphism: that $\exp(a+b) = \exp(a) \cdot \exp(b)$ for every $a, b \in \mathbb{R}$. But this is a well-known property of the exponential.

6. (a) $f(e_G) = e_H$ and $f(a^{-1}) = f(a)^{-1}$ are checked in the exact same way as in class for isomorphisms.

   (b) Since $f(e_G) = e_H$ we have $e_G \in \ker f$, which is non-empty.

   Let $a, b \in \ker f$ (i.e., $f(a) = e_H$, $f(b) = e_H$). We want $a \cdot b \in \ker f$ i.e., $f(a \cdot b) = e_H$. By definition of f we have $f(a \cdot b) = f(a) * f(b) = e_H * e_H = e_H$.

   Let $a \in \ker f$. We want $a^{-1} \in \ker f$ i.e. $f(a^{-1}) = e_H$. But $f(a^{-1}) = f(a)^{-1} = e_H^{-1} = e_H$.