Algebraic Structures (MST20010) 2023/2024

Problem sheet 1

1. Already done in class, look at the course notes. But really do it again on your own for practice.

2. In $\mathbb{Z}/5\mathbb{Z}$: $8^{92} = 3^{92}$. Consider now the sequence of powers of 3 in $\mathbb{Z}/5\mathbb{Z}$: $1, 3, 3^2 = 4, 3^3 = 3^2 3 = 12 = 2, 3^4 = 3^3 3 = 6 = 1, \ldots$ (since we reached 1, if we keep multiplying by 3 we will then get 3, $3^2 =$, etc we are back at the start of the sequence above). We see that $3^k = 1$ if k is a multiple of 4, $3^k = 3$ if k is a multiple of 4 plus 1, $3^k = 4$ if k is a multiple of 4 plus 2, and $3^k = 2$ is k is a multiple of 4 plus 3. So $3^{92} = 1$.

   $(13^{15} \cdot 5^{26}) + 4 \cdot 26^{32} = (3^{15} \cdot 0^{26}) + 4 \cdot 1^{32} = 4$.

3. (a) Since the same day comes back every 7 days, we want to compute the remainder of 47×642 in the division by 7, i.e., 47×642 in $\mathbb{Z}/7\mathbb{Z}$. We have, in $\mathbb{Z}/7\mathbb{Z}$ (using Proposition 1.4): $47 \times 642 = 5 \times 5 = 25 = 4$. So it will be a Friday (Monday plus 4 days).

   (b) Observe that cutting a piece in 7 increases the number of pieces by 6, so the remainder modulo 6 is unchanged, and is equal to $7 \bmod 6 = 1$. But $1997 = 6*300+197 = 6*330+17 = 6*332+5$, so $1997 \bmod 6 = 5$.

4. (a) $2 \cdot 2 = 4 = 1$.

   (b) We compute $3a + 5b$ and 8 in $\mathbb{Z}/3\mathbb{Z}$ (using Proposition 1.4): $3a + 5b = 0a + 2b = 2b$ and $8 = 2$. So, since $3a + 5b = 8$ they have the same remainder in the division by 3, so they are equal in $\mathbb{Z}/3\mathbb{Z}$: $2b = 2$. Multiplying both sides by 2 we get (in $\mathbb{Z}/3Z$): $4b = 4$ and thus $b = 1$ (using the first part).