Problem sheet 0

In both cases we want to solve a simple equation. One way to do this is to rewrite it in several steps, keeping an equality at each step, and ending with $x = \cdots$. One way to keep the equality between each step is to apply the same operation to both sides of the equality. We will do just that, and then we will compare what we did in each case

1.

$$\begin{aligned} x + a &= b && \text{add } -a \text{ to both sides} \\ x + a + (-a) &= b + (-a) && \text{compute } a + (-a) \\ x + 0 &= b - a && \text{compute } x + 0 \\ x &= b - a. \end{aligned}$$

We used that $x + 0 = x$ and that there is an element (called $-a$) with the property that $a + (-a) = 0$.

We also used something that was a bit less visible, in going from line 2 to line 3: The left hand side of line 2 is actually $(x+a)+(-a)$ because we add $-a$ to $x + a$, so the operation $x + a$ comes first. But then we decide that it is ok to compute $a + (-a)$ first. To be able to do this, we are using the property $(x+a)+(-a) = x+(a+(-a))$. In general, we are using the following property of the sum of elements of $\mathbb{R}$

$$(a + b) + c = a + (b + c),$$

which is true for every $a, b, c \in \mathbb{R}$.

2.

$$\begin{aligned} x \cdot a &= b && \text{multiply both sides by } a^{-1} \\ x \cdot a \cdot a^{-1} &= b \cdot a^{-1} && \text{compute } a \cdot a^{-1} \\ x \cdot 1 &= b \cdot a^{-1} && \text{compute } x \cdot 1 \\ x &= b \cdot a^{-1}. \end{aligned}$$

We used that $x \cdot 1 = x$, and that there is an element (called a^{-1}) such that $a \cdot a^{-1} = 1$.

As in the case of the sum, we were a bit too quick in going from line 2 to line 3. The left hand side of line 2 is actually $(x \cdot a) \cdot a^{-1}$, and we decided that it was the same if we compute $a \cdot a^{-1}$ first, i.e., that $(x \cdot a) \cdot a^{-1} = x \cdot (a \cdot a^{-1})$. In fact we used the following property of real numbers:

$$(a \cdot b) \cdot c = a \cdot (b \cdot c),$$

for every $a, b, c \in \mathbb{R}$.

3. Obviously, what we did in both cases looks similar. The only difference is that we had a different name for the operation, and that one remarkable element was called 0 in the first case, and 1 in the second. Let us reformulate it all in a way that will cover both cases.

   We call the operation $*$ (so it can be sum or product). The starting equation is $x * a = b$. In both cases we have an element e with the property that $x * e = x$ (for any x; it is 0 in the first case, and 1 in the second). And in both cases we have an element a' such that $a * a' = e$. Let us use this to solve the equation:

$$\begin{aligned} x * a &= b && \text{apply } *a' \text{ to both sides} \\ x * a * a' &= b * a' && \text{compute } a * a' \\ x * e &= b * a' && \text{compute } x * e \\ x &= b * a'. \end{aligned}$$

   Again, we need to be a bit more precise in line 2: The left hand side is actually $(x * a) * a'$, and we use that it is the same as $x * (a * a')$.

   To sum up, we have used the following properties:

   (a) For every u, v, w: $(u * v) * w = u * (v * w)$;

   (b) There is an element e such that, for every element (let's call it z): $z * e = z$;

   (c) For every element (let us again call it z), there is an element z' such that $z * z' = e$.

   So, if we know this about an operation $*$, we can solve simple equations of the type $x * a = b$ (no need to redo the reasoning each time, what we did just above will cover all cases).