Problem sheet 9

1. (a) Show that $(\mathbb{Z}, +)$ is cyclic, and that the only generators of $(\mathbb{Z}, +)$ are 1 and -1.
   (b) Show that $(\mathbb{R} \setminus \{0\}, \cdot)$ is not cyclic, i.e., that there is no $a \in \mathbb{R} \setminus \{0\}$ such that $\mathbb{R} \setminus \{0\} = \langle a \rangle$. (Only do the case $a > 0$; the other is similar, just a bit longer.)
   (c) Show that $(\mathbb{Z}/n\mathbb{Z}, +)$ is cyclic.
   (d) Find all the generators of $(\mathbb{Z}/3\mathbb{Z}, +)$ (i.e., all the elements $a \in \mathbb{Z}/3\mathbb{Z}$ such that $\mathbb{Z}/3\mathbb{Z} = \langle a \rangle$). Same question for $(\mathbb{Z}/6\mathbb{Z}, +)$.

2. Let G be a group and let $a \in G$. Define

$$C_G(a) = \{x \in G \mid xa = ax\}.$$

   Show that $C_G(a)$ is a subgroup of G.

3. Show that $H = \{\mathrm{id}, (1\ 2)\}$ is a subgroup of S_3. Find $\sigma \in S_3$ such that $H\sigma \neq \sigma H$.

4. Let G be a group and let H be a subgroup of H. Let $a \in G$. Show that $a \in H$ is equivalent to $aH = H$. You can do it either directly from the definition of aH, or using the fact that aH is the equivalence class of a for the equivalence relation $\sim_H$.

5. The objective of this exercise is to show that every subgroup of $(\mathbb{Z}, +)$ is of the form $a\mathbb{Z}$ for some $a \in \mathbb{N} \cup \{0\}$. (It is an easy, but important, result.)

   Let H be a subgroup of $(\mathbb{Z}, +)$, $H \neq \{0\}$. Define a to be the smallest positive element of H (why does it exist?).

   (a) Show that $a\mathbb{Z} \subseteq H$.

   Let $n \in H$, and let $n = aq + r$ be the division of n by a with remainder $r \in \{0, \ldots, a-1\}$.

   (b) Show that $r \in H$.
   (c) Deduce that $r = 0$ and that $n \in a\mathbb{Z}$.
   (d) Conclude.