Algebraic Structures (MST20010) 2023/2024

Problem sheet 8

1. (a) Let $n \in \mathbb{N}$. Show that $n\mathbb{Z} = \{nx \mid x \in \mathbb{Z}\}$ is a subgroup of the group $(\mathbb{Z}, +)$. (Seen in class, will not be covered again in the tutorials.)
   (b) Show that $\mathbb{N} = \{0, 1, 2, 3 \ldots\}$ is not a subgroup of $(\mathbb{Z}, +)$.
   (c) Show that $\{-1, 1\}$ is a subgroup of $(\mathbb{R} \setminus \{0\}, \cdot)$.
   (d) If H and K are subgroups of a group G, show that $H \cap K$ is a subgroup of G.

2. In the group S_4, determine the subgroup generated by the elements (1 2) and (3 4).

3. Let G be a group and let $x \in G$ be of order n. Assume that $n = rs$ for some $r, s \in \mathbb{N}$. Show that x^r has order s.

4. Let G be a group and let H be a subgroup of G. Let $a \in G$. Define

$$aHa^{-1} = \{aha^{-1} \mid h \in H\}.$$

   Show that aHa^{-1} is a subgroup of G.
   (Remark: If G is abelian, then $aha^{-1} = h$, so $aHa^{-1} = H$. In particular, this construction is only realy useful –and it is useful– when G is not abelian.)