Problem sheet 6

1. Determine the following element of S_7: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 3 & 7 & 4 & 6 & 1 & 2 \end{pmatrix}^{-1} (3\,7\,2)$, calculate its order, and determine if it is odd or even.

2. Let $\sim$ be the relation on $\mathbb{R}^2$ defined by $(x, y) \sim (a, b)$ if and only if there is $\alpha \in \mathbb{R}^{>0}$ such that $\alpha \cdot (x, y) = (a, b)$ (recall that $\alpha \cdot (x, y) = (\alpha x, \alpha y)$). Show that $\sim$ is an equivalence relation on $\mathbb{R}^2$ and represent graphically all its equivalence classes.

3. Let G be a group.

   (a) Suppose that for every $a, b, c \in G$ we have
   $$ac = cb \text{ implies } a = b.$$
   Show that G is Abelian i.e., for every $x, y \in G$ $xy = yx$.

   (b) Show that if every element of G has order at most 2 (i.e., $x^2 = e$ for every $x \in G$), then G is Abelian. Hint: Consider the element xy and use that $x^2 = e$, $y^2 = e$.

4. Consider an equilateral triangle with vertices A, B, C. Determine all symmetries of the triangle and write down the multiplication table of the operation $\circ$ between these isometries (the composition of maps). We saw (or will shortly see) in class that this set, together with this operation, is a group denoted D_6, called the dihedral group of order 6.

5. We saw in exercise sheet 5 that if $f : A \to B$ is a function, then the relation on A defined by $x \sim y$ iff $f(x) = f(y)$ is an equivalence relation. This exercise shows that every equivalence relation can be obtained in this way.

   Let $\sim$ be an equivalence relation on a set S. Let $\{A_i\}_{i \in I}$ be the collection of all different equivalence classes of $\sim$ (so $A_i \cap A_j = \emptyset$ if $i \neq j$). Observe that every element of S belongs to an equivalence class (indeed $x \in [x]$, and $[x]$ is one of the A_i). Define the map $f : S \to I$, $f(x) =$ the unique i such that x belongs to A_i.

   Show that $x \sim y$ if and only if $f(x) = f(y)$.