Algebraic Structures (MST20010) 2023/2024

Problem sheet 5

1. For each of the following permutations of S_7, indicate if it is odd or even.

   (a) $(1\ 7\ 4\ 5\ 2)(2\ 4\ 7\ 6)$.

   (b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 6 & 7 & 4 & 5 & 2 & 1 \end{pmatrix}$.

2. Show that the inverse of an even permutation is even, and the product of an odd and an even permutation is odd.

3. Show that there is no solution in S_n to the equation

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \sigma = \tau$$

   with σ cycle of length 4 and τ even (think of the parity of the permutations).

4. (a) Let S be the set of all UCD students and B the relation "having the same birthday", i.e. if x and y are students, xBy means that x and y have the same birthday. Show that B is an equivalence relation on S.

   (b) Let $f : A \to B$ be a function. We define a relation R on A by

$$xRy \Leftrightarrow f(x) = f(y).$$

   Show that R is an equivalence relation on A.

5. We define a relation on $\mathbb{Z}$ by: xRy if and only if x and y have a common divisor greater than 1. Show that R is not an equivalence relation.

The following exercise is for you to practice computing with permutations (if you have not done so already, do practice it, it is important. Just compute a few random products –also of more than 2 permutations–, inverses, how to write a permutation as a product of disjoint cycles, how to determine the order, the parity). It will not be corrected in the tutorials, the solution is upside-down just after it. Ask me in class if you want more explanations or had difficulties. (Do ask! It is very important!)

1. (a) Compute $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 1 & 7 & 5 & 2 & 6 & 3 \end{pmatrix}^{-1}$, write is as a product of disjoint cycles, and compute its order (hint: having it writen as a product of disjoint cycles is useful for this).

(b) Compute $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}$, and its order.

(c) Determine the following permutation of S_6: $(1\ 2)(2\ 3)(1\ 4)$ (i.e., write is in the form of a table with two lines that we use to represent permutations)..

Answer 1 *(a)* $= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 5 & 7 & 1 & 4 & 6 & 3 \end{pmatrix} = (1\ 2\ 5\ 4)(3\ 7)$. *Its order is* $lcm(4, 2) = 4$.

(b) $= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = (1\ 4)(2\ 3)$ *so its order is 2.*

(c) $= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 1 & 2 & 5 & 6 \end{pmatrix}$.