Problem sheet 3

1. Let G be a group and let $a \in G$.

   (a) Show that if $a^2 = e$ then $a^{-2} = e$.

   (b) More generally, show that if $k \in \mathbb{N}$ and $a^k = e$, then $a^{-k} = e$.

   (c) Deduce that $|a| = |a^{-1}|$, (Recall that $|a|$ denotes the order of a.)

2. Which of the following are groups, and why?

$$(\{-1, 1\}, \cdot), \text{ where } \cdot \text{ is the usual product of integers.}$$

$$(\mathbb{N} \cup \{0\}, +).$$

3. Write down all the elements of S_3 in the form

$$\begin{pmatrix} 1 & 2 & 3 \\ a & b & c \end{pmatrix}.$$

4. Consider the following elements of S_4:

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

   (a) Compute $\alpha\beta$ and $\beta\alpha$.

   (b) Find the smallest integer k such that $\alpha^k = \mathrm{Id}$.

   (c) Show that $\alpha^{k-1} = \alpha^{-1}$.

   (d) Determine β^{-1}.

5. We saw on an example that the product of two permutations in S_4 is in general not commutative.

   (a) Describe all the elements of S_2 and show that the product of elements of S_2 is always commutative (i.e. $\sigma\gamma = \gamma\sigma$ for every $\sigma, \gamma \in S_2$).

   (b) Show that the product of elements in S_n is in general not commutative (i.e. that there are least 2 elements α, β of S_n such that $\alpha\beta \neq \beta\alpha$) whenever $n \geq 3$.